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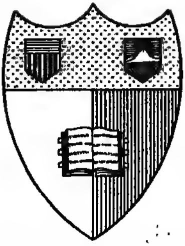

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PROJECTIVE GEOMETRY

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PROJECTIVE GEOMETRY

FOR

USE IN COLLEGES AND SCHOOLS

BY

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PREFACE

THE present work on Projective Geometry has been written as a companion volume to my former treatise on *Homogeneous Coordinates.* It deals, from a geometrical point of view, with those subjects which, in the former volume, were treated analytically. In writing the book I have kept in view the needs of students reading for honours at the Universities and of pupils specialising in Mathematics at the Secondary Schools. I have assumed on the part of the reader a competent knowledge of the Modern Geometry of the Triangle and Circle, and also of Geometrical Conics; but for the sake of clearness and precision I have quoted the main results drawn from these subjects in black type *without proof*, and have given references, when necessary, where demonstrations may be found. In this way a good deal of ground has been covered, and the book has not been made too long. On the other hand, no attempt has been made to treat of Homography and Involution, as I feel that the student requires a fairly complete account of them before he can make any use of them as instruments of geometrical investigation, and such detailed treatment would have increased the size of the book unduly.

In teaching Projective Geometry, it is advisable to do a considerable amount of oral instruction with the aid of the blackboard, in order that the students may gain a comprehensive view of the subject apart from its details. *Viva voce* practice in working out the theory of general subjects, such as Reciprocation, and in analysing and synthesising general theorems, such as the general properties of Four-Point and Four-Line Conics, is of the utmost value. As other examples of the same kind one may suggest the general properties of five lines that flow from the fact that they uniquely define a conic, and those of four lines that come from the fact that they uniquely define a parabola. These, and such-like exercises, form a fitting preparation for the essay papers on general subjects now set in the College and University Examinations. Projective Geometry lends itself naturally and effectively to instruction by this method, in addition to the practice afforded by the indiscriminate solution of disconnected problems.

Chapter I. deals with the method of Projection applied to figures involving Straight Lines only. In Chapter II. the Conic is introduced, and there is a very full treatment of the uses and properties of the Circular Points at Infinity. The discussion depends on elementary considerations only, and it is hoped that the beginner will find this chapter easy reading, as it seems a pity that he should postpone such simple, elegant, and powerful methods of geometrical investigation until he has first mastered the properties of Involutions. On the other hand, the theoretical and technical difficulties of the subject have been dealt with in detail, and the Principle of Continuity in Geometry has been explained at considerable length. Chapter III. is devoted to Reciprocation, and it is hoped that the frequent illustrations and examples will do much to clear away the difficulties with which this subject is usually beset for the beginner. Chapter IV. contains a fairly full account of the main properties of Conics in so far as they can be dealt with by the methods of projection only. I have made an innovation by calling "Desargues's Theorem" the general theorem, of which his is only a particular case. Much convenience is thereby gained.

The examples have for the most part been selected and arranged from the Scholarship Papers set by the various Colleges of Cambridge, and I have to thank the Syndics of the University Press and the various College Tutors for their kind permission to make use of them.

I wish also to express my obligations to Mr. J. H. Grace, F.R.S., formerly Fellow of St. Peter's College, Cambridge; Mr. H. C. Beaven, M.A., Head of the Mathematical Department, Clifton College; Mr. Peter Fraser, M.A., B.Sc., Lecturer in Mathematics at the University of Bristol; Mr. F. C. Stephen, M.A., Emmanuel College, Cambridge, for kindly criticising the work while in manuscript form and giving many valuable suggestions. I am also greatly indebted to Mr. Charles M'Leod, M.A., D.Sc., Senior Mathematical Master, Aberdeen Grammar School, who not only read the work while in manuscript, but also revised the proofs while it passed through the press.

I shall be grateful for any corrections or suggestions from those who use the book.

W. P. M.

CLIFTON COLLEGE,
July 1911.

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EXPLANATORY NOTE

In the following list of references to the geometrical theorems stated without proof in black type throughout the text, the abbreviation *Seq. to Euc.* refers to the work entitled "A Sequel to the First Six Books of the Elements of Euclid," by John Casey, LL.D., F.R.S.; *Seq. to El. Geom.* to "A Sequel to Elementary Geometry," by John Wellesley Russell, M.A.; *Geom. Con.* to "Geometrical Conics," by Charles Smith, M.A.; *El. Geom. of Con.* to "The Elementary Geometry of Conics," by C. Taylor, D.D.

4. *Seq. to Euc.* Bk. vi. Sect. vi. Prop. 1; *Geom. Con.* Chap. vii. Art. 151.
8. *Seq. to Euc.* Bk. vi. Sect. i. Prop. 2; *Seq. to El. Geom.* Chap. iii. Art. 14.
9. *School Geometry*, by Hall and Stevens, Theorem 68.
15. *Seq. to Euc.* Bk. vi. Sect. iii. Prop. 4; *Seq. to El. Geom.* Chap. vi. Art. 7; *School Geometry*, by Hall and Stevens, page 334, Ex. 4.
23. *Seq. to Euc.* Bk. vi. Sect. v. Prop. 2, Cor. 5; *Seq. to El. Geom.* Chap. vii. Art. 6.
35. *Geom. Con.* Chap. iii. Prop. 7; *El. Geom. of Con.* Chap. iv. Prop. 7.
36. *Geom. Con.* Chap. ii. Prop. 9; *El. Geom. of Con.* Chap. iii. Prop. 8.
37. *Geom. Con.* Chap. iii. Prop. 7; *El. Geom. of Con.* Chap. iv. Prop. 7.
41. *Seq. to Euc.* Bk. vi. Sect. iv. Prop. 15.
42. *Geom. Con.* Chap. ii. Prop. 12; *El. Geom. of Con.* Chap. iii. Prop. 11.
43. *Seq. to Euc.* Bk. vi. Sect. v. Prop. 1 (4); *Seq. to El. Geom.* Chap. vii. Art. 14.

CHAPTER I.

THE STRAIGHT LINE.

1. *Projective Geometry.*

Our object is to deal with the more elementary geometrical properties that are not altered by the process of "*projection.*" This process is explained in Art. 3.

In this way we shall shew that many properties of elementary Euclidean geometry (*e.g.* of circles) lead at once to properties of higher geometry (*e.g.* of conics); and, conversely, that many general theorems on conics, etc., can be proved *after projection* by quite elementary methods.

2. *Notation.*

We shall denote *points* by capital letters, **A**, **B**, **C**, etc.
„ „ *lines* „ small „ a, b, c, etc.
„ „ *planes* „ Greek „ α, β, γ, etc.

AB will denote the line joining the points **A** and **B**.

ab will denote the point of intersection of the lines a and b.

Aa will denote the plane containing the point **A** and the line a.

3. *Method of Projection.*

Let π and π' be two given fixed planes and O a given fixed point.

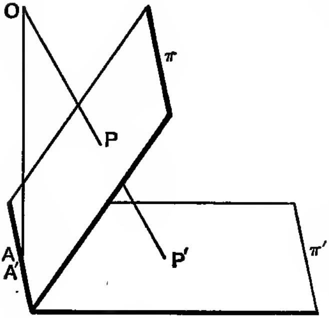

FIG. 1.

Let P be any point in the plane π. Join O to P meeting π' in the point P'.

P' *is called the projection of the point* P *on the plane* π' *with respect to the Vertex of Projection* O.

COROLLARY I. P *is the projection of* P' *on the plane* π.

COROLLARY II. *Any point on the line of intersection of the planes* π *and* π' *projects into itself.*

This is obvious from Fig. 1, where the points A and A' are seen to coincide.

4. *Projection of Figures.*

If a figure be traced on the plane π, and if every point on this figure be projected with respect to the Vertex of Projection O on the plane π', a new figure will be obtained.

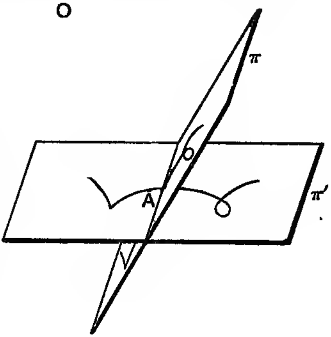

FIG. 2.

The figure thus obtained on the plane π' *is called* *the "Projection of the figure drawn on the plane* π *with respect to the Vertex of Projection* O."

The diagram represents a characteristic figure in the Higher Geometry of Curves projected from the plane π on to the plane π' (or *vice versa*), O being the vertex of projection.

5. *The projection of a straight line is a straight line.*

Let a given straight line l lie in the plane π, and let a plane λ be passed through O and the line l; let this plane λ cut the plane π' in the line l'. Then, if we join O to any point P in the line l, it is plain that OP will cut π' in some point P′ lying on l'. Hence the straight line l' is the projection of the straight line l with respect to the vertex of projection O.

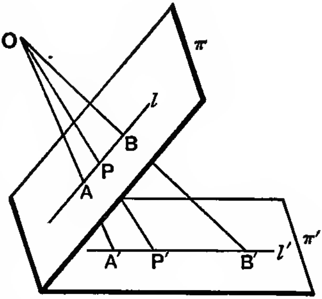

FIG. 3.

RANGE OF POINTS. If a number of points all lie on the same straight line, they are called a range of points.⁽¹⁾

COROLLARY I. *The projection of a range of points is a range of points.*

COROLLARY II. *The projection of two intersecting straight lines is two intersecting straight lines.*

For, a point of intersection of any two curves lying in π projects into a point of intersection of the two projected curves lying in π'.

PENCIL OF LINES. If a number of straight lines all lie in the same plane and all pass through the same point, they are said to form a pencil of lines.(2)

COROLLARY III. *The projection of a pencil of lines is a pencil of lines.*

6. The following propositions will now be required.

CROSS-RATIO OF FOUR COLLINEAR POINTS. If P, Q, R, S be four collinear points, their cross-ratio, usually denoted by [PQRS], is defined as $\frac{PQ \cdot RS}{PS \cdot RQ}$, the usual convention as to the signs of the various segments of the given line being adopted.(3)

The following method of remembering the order of the letters in the above definition of cross-ratio will be found convenient.

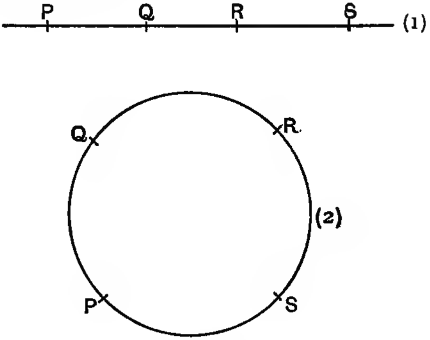

FIG. 4.

Let P, Q, R, S be the four points in this order, as in (1) Fig. 4. Imagine them placed round a circle as in (2). For the numerator of the cross ratio, write down the letters in clockwise order, and for the denominator write them down in counter-clockwise order, beginning with the same letter in both cases.

Cross-Ratio of a Pencil of four Concurrent Lines. If four coplanar lines OA, OB, OC, OD meet in O, and if two transversals cut them in the four points P, Q, R, S and P′, Q′, R′, S′ respectively, then will $[PQRS]=[P'Q'R'S']$, *i.e.* all transversals are cut by the same pencil of four lines in the same cross-ratio.(4)

Cross-Ratio of a Pencil of four Concurrent Lines, in terms of the Angles between the Rays. If four coplanar lines OA, OB, OC, OD meet in O, and if a transversal cut them in the four points A, B, C, D respectively, then the cross-ratio $[ABCD]$ is equal to $\frac{\sin AOB \sin COD}{\sin AOD \sin COB}$, the usual convention being adopted as to the signs of the various segments of the given angle AOD.(5)

Cross-Ratio of two Pencils in which the Angles between the Corresponding Constituent Rays are Equal, each to each. If OA, OB, OC, OD and O′A′, O′B′, O′C′, O′D′ be two pencils of rays in which $\angle AOB=\angle A'O'B'$; $\angle COD=\angle C'O'D'$; $\angle AOD=\angle A'O'D'$; $\angle COB=\angle C'O'B'$; then it is plain from last paragraph that the cross-ratio $O[ABCD]$ = the cross-ratio $O'[A'B'C'D']$.(6)

Harmonic Division. The two points P and Q are said to harmonically separate R and S if

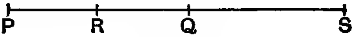

Fig. 5.

$$\frac{PR}{RQ}+\frac{PS}{SQ}=0,$$

i.e. if $[PRQS]$ has the particular value -1.

Also P and Q are called **Harmonic Conjugates** with respect to R and S.(7)

7. *The Cross-Ratio of Four Collinear Points is unaltered by projection.*

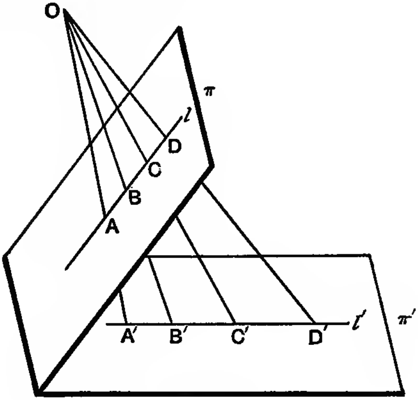

FIG. 6.

Let O be the vertex of projection and l a line lying in the plane π. Let a plane λ passing through O and l cut the plane π' in the line l'. Let A′, B′, C′, D′ be the four points on l' that are respectively the projections of the four points A, B, C, D lying on l (Chap. I. Art. 5).

Then by Chap. I. Art 6 [ABCD]=[A′B′C′D′], which proves the proposition.

8. *The Line at Infinity in a given plane.*

Consider first of all a point O and a line p where p does not pass through O, O and p being not entirely at infinity. Let P be a point on the line p.

Then as P moves away to infinity on p, the line OP tends to become more and more nearly parallel to p. We therefore say that if P lies at infinity on p, then OP is parallel to p.

Furthermore, since one and only one line can be drawn through **O** parallel to p, we see that *we must regard the line p in Projective Geometry as containing only one point at infinity,* provided that p itself does not lie entirely at infinity.

Consider next a point **O** and a plane π where π does not pass through **O**, **O** and π being not entirely at infinity.

We note first of all that we can draw through **O** one and only one plane λ parallel to π. This can easily be deduced from Playfair's Axiom on parallels. Also all lines drawn through **O** parallel to π lie in λ. This can also be easily proved by elementary geometry. Let **P** be a point in the plane π.

Then as **P** moves away to infinity on π, the line **OP** tends to become more and more nearly parallel to π.

We therefore say that if **P** lies at infinity on π, then **OP** is parallel to π.

But if **OP** is parallel to π, then **OP** lies in the unique plane λ; that is, **P** is a point common to the planes π and λ.

Hence every point at infinity in π must be regarded in Projective Geometry as lying also in λ.

But two planes in general intersect in a straight line, hence two parallel planes must be regarded as intersecting in a straight line infinitely far away.

Thus, by what precedes, every point at infinity on π must be regarded in Projective Geometry as lying on the straight line $\pi\lambda$ infinitely far away.

The unique straight line on which all the points at infinity in a given plane must be regarded as lying *in Projective Geometry, is called the Line at Infinity in that plane.*

NOTE I. *Two parallel lines l and m lying in the plane π must be regarded as having their point of intersection lm situated on the Line at Infinity in π.*

NOTE II. *If l_1 and m_1 be two parallel lines lying in π, and l_2 and m_2 two other parallel lines lying in π, but such that l_1, etc., is not parallel to l_2, etc., then must l_1m_1 and l_2m_2 be regarded as two separate and distinct points lying on the Line at Infinity in π.*

9. *If l be a given line in a given plane π, and O be a given vertex of projection, to find a plane π' such that the projection of l on to π' may be the Line at Infinity on π'.*

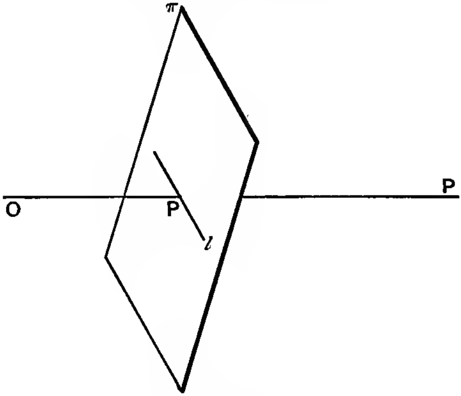

FIG. 7.

Choose as π' a plane parallel to the plane Ol (*i.e.* the plane containing the point O and the line l). Then it is plain that the line joining O to any point P on l will be parallel to the plane π', *i.e.* will meet π' at infinity. Hence, if P' be the point at infinity in which OP meets the plane π', P' will lie on the Line at Infinity in π'.

Thus if π' be chosen parallel to the plane Ol, the projection of l lying in π will be the Line at Infinity lying in π'.

COROLLARY I. *The projection of the Line at Infinity lying in* π *will be the intersection of the plane through* O *parallel to* π *with the plane* π'.

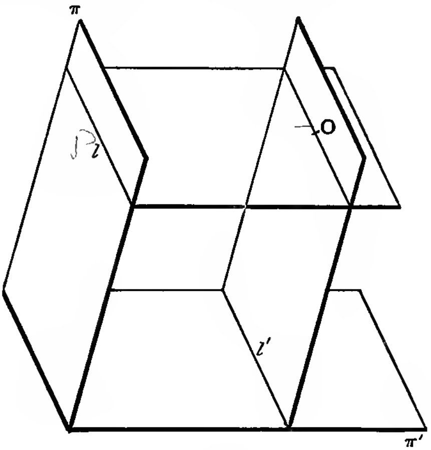

FIG. 8.

In Fig. 8, l is the projection on to the plane π of the Line at Infinity on π'.

In Fig. 8, l' is the projection on to the plane π' of the Line at Infinity on π.

COROLLARY II. *Three points* A', B', C' *lying on the Line at Infinity in* π' *project into three collinear points* A, B, C *lying in the finite part of* π.

10. *To project a given line to infinity.*

The operation of selecting a suitable vertex of projection O and plane of projection π' so that a given line situated on a given plane π shall appear as the Line at Infinity on π' is called the operation of "projecting a given line to infinity."

11. Many geometrical theorems which would be very hard to solve if attempted directly, become quite simple if attacked by the methods of Projective Geometry.

We give some examples.

Example 1.

The three lines UP_1P_2, UQ_1Q_2, UR_1R_2 *meet the two lines* OX_1 *and* OX_2 *in* P_1, Q_1, R_1 *and* P_2, Q_2, R_2 *respectively. Prove that the intersections of* Q_1R_2, Q_2R_1 *and* R_1P_2, R_2P_1 *and* P_1Q_2, P_2Q_1 *are collinear.*

Project the line OU to infinity.

Let, as usual, O', U', etc., denote the projections of O, U, etc. Draw the projected figure directly from the enunciation. We thus obtain Fig. 9.

It is thus plain that the intersections of $Q_1'R_2'$, $Q_2'R_1'$ and $R_1'P_2'$, $R_2'P_1'$, and $P_1'Q_2'$, $P_2'Q_1'$ are collinear (since they lie on the line lying mid-way between $O'X_1'$ and $O'X_2'$). Hence, in the original figure the corresponding points are collinear (Chap. I. Art. 5, Cor. I.).

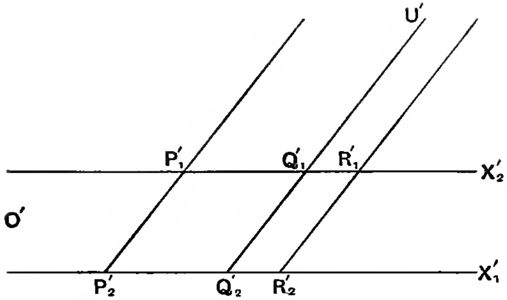

FIG. 9.

Example 2.

We shall require Ceva's Theorem to solve the present example:

CEVA'S THEOREM. If ABC be a triangle and O any point, and if the lines joining O to the vertices meet the opposite sides in D, E, F respectively, then

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1.^{(8)}$$

If in the figure to Ceva's Theorem, EF *meet* BC *in* L, FD *meet* CA *in* M *and* DE *meet* AB *in* N, *then* L, M, N *lie on the same straight line (called the Polar Line of* O *with respect to the triangle* ABC*).*

Project LM into the Line at Infinity.

Then since F′D′ and E′F′ are parallel to C′A′ and B′C′ respectively, $\therefore\ \frac{A'F'}{F'B'}=\frac{C'D'}{D'B'}=\frac{A'E'}{E'C'}$. ....(1)

But by Ceva's Theorem

$$\frac{A'F'}{F'B'}\cdot\frac{B'D'}{D'C'}\cdot\frac{C'E'}{E'A'}=1. \quad ....(2)$$

Hence by (1) and (2) $A'F' = F'B'$,
$B'D' = D'C'$,
$C'E' = E'A'$;

$\therefore$ $D'E'$ is parallel to $A'B'$.

i.e. N' lies on the Line at Infinity $L'M'$;

$\therefore$ L, M, N are collinear.

COROLLARY. *We can project a given triangle into another triangle having the projection of a given point* O *as its Centroid, by projecting the Polar Line of* O *with respect to the original triangle into the Line at Infinity.*

This corollary is often very useful.

Example 3.

The following proposition will be required for the solution of the present example:

SIMILAR AND SIMILARLY SITUATED TRIANGLES. If two triangles be such that the sides of the one are parallel to the sides of the other, each to each, then will the three lines joining corresponding vertices be concurrent.(9)

If $A_1B_1C_1$ *and* $A_2B_2C_2$ *be two triangles such that*
B_1C_1 *and* B_2C_2 *meet in* L,
C_1A_1 *and* C_2A_2 *meet in* M,
A_1B_1 *and* A_2B_2 *meet in* N,
where LMN *are collinear, then will* A_1A_2, B_1B_2, C_1C_2 *be concurrent.*

For project LMN into $L'M'N'$, the line at infinity. Then the two triangles $A_1'B_1'C_1'$ and $A_2'B_2'C_2'$ will be such that corresponding sides are parallel.

Hence $A_1'A_2'$, $B_1'B_2'$, $C_1'C_2'$ are concurrent (by proposition just quoted), $\therefore$ A_1A_2, B_1B_2, C_1C_2 are concurrent (Chap. I. Art. 5, Cor. III.).

TRIANGLES IN PERSPECTIVE. Two triangles, such as $A_1B_1C_1$ and $A_2B_2C_2$, are said to be "in perspective."

The meeting point of the lines A_1A_2, B_1B_2, C_1C_2 is called the "centre of perspective."

The line LMN is called the "axis of perspective."(10)

Example 4.

If a fixed line p, lying in the plane π, be projected from an arbitrary vertex of projection O into the line p' lying in the plane π', p' passes through a fixed point.

Let the line p cut the intersection of π and π' in A. Then plainly p' passes through A, which is a fixed point.

12. *Cross-ratio of four points, one of which is at infinity.*

FIG 10.

Let the point D be at infinity, A, B, C being supposed in the finite part of the plane.

Then $[ABCD] = \frac{AB . CD}{AD . CB}$ (Chap. I. Art. 6). ....(1)

But if D be at infinity, $\frac{AD}{CD} = 1$ in the limit.

Hence, by (1), $[ABCD] = \frac{AB}{CB}$.

Hence, when one of the four points is at infinity, the cross-ratio of the four points becomes the ratio of two finite segments formed by the three points in the finite part of the plane.

COROLLARY. *If* B *and* D *harmonically separate* A *and* C, *and if* D *be at infinity,* B *is the middle point of* AC.

For by Chap. I. Art. 6,

$$[\mathrm{ABCD}] = -1,$$

i.e.
$$\frac{\mathrm{AB}\,.\,\mathrm{CD}}{\mathrm{AD}\,.\,\mathrm{CB}} = -1,$$

and since $\frac{\mathrm{CD}}{\mathrm{AD}} = 1$ in the limit when D is at infinity,

$$\therefore\ \frac{\mathrm{AB}}{\mathrm{CB}} = -1$$

i.e. B is the mid-point of AC.

13. *The Cross-ratios of four given Collinear Points.*

Let ABCD be four given collinear points. We can form, from these, twenty-four cross-ratios corresponding to the $\underline{|4} = 24$ possible permutations of A, B, C, D. We can shew that these 24 possible cross-ratios are not all different.

Take any line XY (other than the line ABCD) through D, and project XY to infinity.

Let A′, etc., be the projection of A, etc.

Then, taking all the cross-ratios in which D appears last, we get, remembering that D′ is at infinity,

$$[\mathrm{ABCD}] = [\mathrm{A'B'C'D'}] = \frac{\mathrm{A'B'}}{\mathrm{C'B'}} = \rho.$$

$$[\mathrm{ACBD}] = [\mathrm{A'C'B'D'}] = \frac{\mathrm{A'C'}}{\mathrm{B'C'}} = \frac{\mathrm{A'B'} + \mathrm{B'C'}}{\mathrm{B'C'}} = 1 - \rho.$$

$$[\mathrm{BCAD}] = [\mathrm{B'C'A'D'}] = \frac{\mathrm{B'C'}}{\mathrm{A'C'}} = \frac{1}{1-\rho}.$$

$$[\mathrm{BACD}] = [\mathrm{B'A'C'D'}] = \frac{\mathrm{B'A'}}{\mathrm{C'A'}} = \frac{\rho}{\rho - 1}.$$

$$[\mathrm{CABD}]=[\mathrm{C'A'B'D'}]=\frac{\mathrm{C'A'}}{\mathrm{B'A'}}=\frac{\rho-1}{\rho}.$$

$$[\mathrm{CBAD}]=[\mathrm{C'B'A'D'}]=\frac{\mathrm{C'B'}}{\mathrm{A'B'}}=\frac{1}{\rho}.$$

Next let us take the set of *six* cross-ratios in which A appears last, and compare with the above.

$$[\mathrm{BCDA}]=[\mathrm{B'C'D'A'}]=\frac{\mathrm{B'C'}}{\mathrm{B'A'}}=\frac{1}{\rho}.$$

$$[\mathrm{BDCA}]=[\mathrm{B'D'C'A'}]=\frac{\mathrm{C'A'}}{\mathrm{B'A'}}=\frac{\rho-1}{\rho}.$$

$$[\mathrm{CDBA}]=[\mathrm{C'D'B'A'}]=\frac{\mathrm{B'A'}}{\mathrm{C'A'}}=\frac{\rho}{\rho-1}.$$

$$[\mathrm{CBDA}]=[\mathrm{C'B'D'A'}]=\frac{\mathrm{C'B'}}{\mathrm{C'A'}}=\frac{1}{1-\rho}.$$

$$[\mathrm{DBCA}]=[\mathrm{D'B'C'A'}]=\frac{\mathrm{C'A'}}{\mathrm{C'B'}}=1-\rho.$$

$$[\mathrm{DCBA}]=[\mathrm{D'C'B'A'}]=\frac{\mathrm{B'A'}}{\mathrm{B'C'}}=\rho.$$

We may deal in the same way with those arrangements in which B and C come last, and we shall find that the 24 possible cross-ratios divide themselves into four groups of six, whose values are

$$\rho,\quad \rho^{-1},\quad 1-\rho,\quad 1-\rho^{-1},\quad \frac{1}{1-\rho},\quad \frac{1}{1-\rho^{-1}},$$

where ρ is the value of the cross-ratio corresponding to any given permutation.

Corollary. *When the range is harmonic the cross-ratio values are* -1, -1, 2, 2, $\frac{1}{2}$, $\frac{1}{2}$.

14. The Complete Quadrangle. If P, Q, R, S be four given points, and if PQ, RS meet in A; PR, SQ in B; PS,

QR in C; the figure so obtained is called the Complete Quadrangle, having A, B, C for vertices.

ABC is sometimes called the "Harmonic Triangle" of the given Quadrangle.(11)

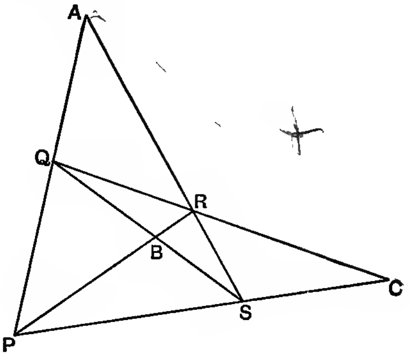

FIG. 11

Harmonic Properties of the Complete Quadrangle.

Project the line AC to infinity into A′C′.

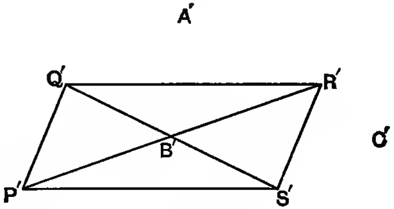

FIG. 12.

Then, since A′ is at infinity,

P′Q′ is parallel to S′R′.

Similarly, Q′R′ „ „ „ P′S′

Hence, P′Q′R′S′ is a parallelogram.

The Harmonic Properties of the Complete Quadrangle can now be read off from Fig. 12, and we shall quote a few by way of illustration.

1°. Let in the original figure PBR meet AC in X. Then

$$[PBRX]=[P'B'R'X']=\frac{P'B'}{R'B'} \text{ (since } X' \text{ is at infinity)}$$
$$=-1$$

Hence, $[PBRX]$ forms a harmonic range.

2°. $A[PBSC]$ *is a Harmonic Pencil.*

For $A[PBSC]=A'['P'B'S'C']$. (Fig. 13.)

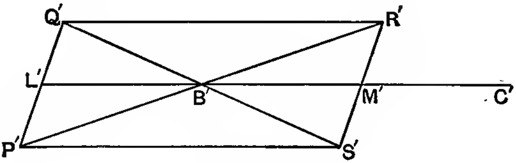

FIG. 13.

But a transversal across the pencil $A'[P'B'S'C']$ is $L'B'M'C'$.

$$\therefore\ A[PBSC]=A'[P'B'S'C']$$
$$=[L'B'M'C']$$
$$=\frac{L'B'}{M'B'} \text{ (since } C' \text{ is at infinity),}$$
$$=-1$$

$\therefore$ $A[PBSC]$ is harmonic (Chap. I. Art. 6).

15. THE COMPLETE QUADRILATERAL. If p, q, r, s be four given lines, and if a be the line joining pq and rs; b the line joining pr and sq; c the line joining ps and qr; the figure so obtained is called the Complete Quadrilateral, and abc is called the Diagonal Triangle.(12)

N.B. The distinction between a "quadrangle" and a "quadrilateral" is often not strictly adhered to in mathematical writings.

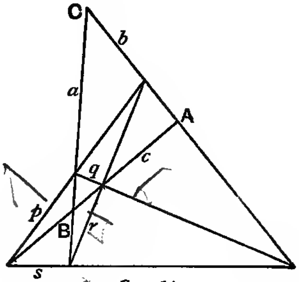

FIG. 14.

16. In solving problems by the methods of Projective Geometry, the reader will probably find it easier not to draw the projected figure from the original figure, as the different appearance to the eye is apt to lead to confusion. Having decided from the original figure what line is to be projected to infinity, he will probably find it simplest *to draw the projected figure directly from the enunciation.*

EXAMPLES I.

1. OX and OY are two fixed straight lines and A is a fixed point. Rays AP_1Q_1, AP_2Q_2, AP_3Q_3, etc., are drawn through A meeting OX and OY in P_1 and Q_1, P_2 and Q_2, P_3 and Q_3, etc., respectively. The intersections of P_1Q_2 and P_2Q_1 and all such similar intersections lie on a straight line OL passing through O. Prove that OL and OA are harmonic conjugates with respect to OX and OY.

2. A and B are two fixed points and XY is a fixed straight line. A series of points P_1, P_2, P_3, ... are taken on XY. L is the intersection of AP_1 and BP_2. M is the intersection of AP_2 and BP_1. Prove that LM passes through a fixed point H on the line AB.

3. If in question (2) XY meet AB in K, shew that [KABH] is harmonic.

4. ABCD is a quadrilateral and XY is a fixed straight line. AB meets XY in P, and P′ is the harmonic conjugate of P with respect to A and B. Q, Q′; R, R′; S, S′ are determined on BC, CD, DA respectively in the same way. Prove that the two points where P′R′ meets Q′S′ and XY are harmonic conjugates with respect to P′ and R′.

5. If the three sides of a triangle always pass through three fixed collinear points respectively, and if two of the vertices move on two fixed straight lines, the third vertex will likewise move on a straight line passing through the intersection of the other two fixed straight lines.

6. OX, OY, OZ are three fixed straight lines. A and B are two fixed points whose join passes through O. R is a variable point on OZ, and RA, RB cut OX and OY in P and Q respectively. Shew that PQ passes through a fixed point U on AB.

7. In question (6), if BR meet OY in Q and OX in Q′, and if AR meet OX in P and OY in P′, then if PQ and P′Q′ meet in L, RL will always pass through the same fixed point H on AB.

8. Let U and U′ be the fixed points in which PQ and P′Q′ of questions (6) and (7) cut AB. Shew that [AOBH] and [UOU′H] are each harmonic.

9. X and Y are two points on the line XYS. Three rays XP, XQ, XR emerge from X and three rays YP, YQ, YR emerge from Y such that X[PQRS] = Y[PQRS]. Prove that the points of intersection P, Q, R are collinear.

10. PQRS is a complete quadrangle having ABC as the vertices of its harmonic triangle, lying on PQ, PR, PS respectively. AB cuts QR in V and PS in U. Shew that PV and UR meet in AC.

11. In question 10, if PV and RU cut QS in E and F respectively, shew that the harmonic conjugate of E with respect to Q and F is the point in which QS cuts AC, and also that this point is the harmonic conjugate of F with respect to E and S.

12. If three points XYZ are taken on the sides BC, CA, AB respectively of a triangle ABC such that AX, BY, CZ are concurrent, and if XY meets BA produced in Z′ and XZ meets CA produced in Y′, prove that AX, BY′, CZ′ are concurrent. (Queens'.)

13. A point P, capable of moving along a given straight line, is joined to two fixed points B, C, and the lines PB, PC intersect in X, Y, another given line. Prove that the locus of the point of intersection of BY and CX is a straight line.

14. PQRS are the four corners of a complete quadrangle having A, B, C, the vertices of the harmonic triangle, lying on PQ, PR, PS respectively. BC meets PQ in H, AB meets QR in K, and PR meets AC in L. Prove that HKL are collinear.

15. PQRS is a complete quadrangle having the vertices ABC of the harmonic triangle lying on PQ, PR, PS respectively. AB meets QR in Y and PS in X. If PY and XR meet in U, and if XQ and SY meet in V, prove that U and V divide AC harmonically.

16. If *pqrs* is a complete quadrilateral having *abc* as its diagonal triangle; *a*, *b*, *c* passing through the points *pq*, *pr*, *ps* respectively; *x* is the join of *ps* and *ab*, while *y* is the join of *qr* and *ab*. Prove that the points *py*, *xr* and *ac* are collinear.

17. PQRS is a complete quadrangle whose vertices are A, B, C lying on PQ, PR, PS respectively. AB meets PS in X and QR in Y, while BC meets PQ in L and RS in M. Shew that two of the diagonals of the quadrilateral formed by the lines PY, QM, RX, SL meet in B.

18. O is a fixed point and XY is a fixed line. P moves on a fixed line LM, and Q is taken on OP such that the range [OPRQ] has a constant cross-ratio, R being the point where OP meets XY. Shew that Q also describes a straight line.

19. If two opposite corners of a quadrilateral and the intersection of their join and one other diagonal are given, prove that the third diagonal passes through a fixed point. (St. John's.)

20. OABC, OA′B′C′ are two straight lines such that AA′, BB′, CC′ are parallel. AB′, A′C meet in P; A′B and AC′ meet in Q. Shew that PQ is parallel to AA′. (Magdalene.)

21. If through O the intersection of the diagonals of a quadrilateral ABCD, a line OH be drawn parallel to the side AB meeting CD in G and the third diagonal in H, prove that OH is bisected at G. (Corpus Christi, etc.)

22. Prove that if two triangles ABC, A′B′C′, in different planes, are such that AA′, BB′, CC′ meet in a point, then the pairs of corresponding sides such as BC, B′C′ meet in three collinear points.

23. Shew that if a triangle PQR be projected from two points O, O′ not lying in its plane, into two triangles ABC, A′B′C′ in a plane not passing through any of the points P, Q, R, O, O′, then these two triangles are in perspective. (Clare, etc.)

24. If the sides of a triangle ABC are divided in D, E, F so that

$$BD : DC = CE : EA = AF : FB,$$

prove that the lines AD, BE, CF can only be concurrent for one particular value of this ratio, and find what that value is. (Magdalene.)

25. Shew that if two coplanar triangles are in perspective and a vertex of one lies on a non-corresponding side of the other (*e.g.* A′ on AB), then, provided the centre of perspective is not at B, a pair of corresponding sides must coincide. Examine the positions of the centre and axis of perspective. (Clare.)

26. The lines AD and BC intersect in C, on BC any point E is taken: AE and BD intersect in Q; CQ and ED intersect in R; BR intersects AD in P. Prove that the position of P is unchanged as E moves on BC (Trinity.)

27. Given a triangle, describe a quadrilateral that shall have the given triangle for its diagonal triangle, and shew that there is one such quadrilateral associated with every point in the plane of the triangle. (St. John's)

28. ABCD, AECF are parallelograms such that EF is parallel to AD. If BE, CF meet in H, and AE, DF in K, shew that HK is parallel to AB. (St. Catharine's.)

29. A and B are two points through which pass respectively the two triads of lines a_1, a_2, a_3, and b_1, b_2, b_3. Shew that the lines joining the pairs of points a_1b_3 and a_3b_2; a_2b_3 and a_3b_1; a_1b_1 and a_2b_2 are concurrent.

30. ABCD is a quadrilateral. M is any point on BC and N any point on AD. On AB, AC, BD, CD are taken points P, Q, R, S respectively so that MQP, MRS are straight lines. If PNR is a straight line, shew that QNS is also a straight line. (Math. Trip.)

31. The corners of a parallelogram, taken in order, are ABCD; E is any point on AB, and DE, CE cut CB, DA in G, F. The parallelograms on DE, EF and CE, EG as sides are completed. Prove that the two diagonals of these parallelograms lie along the same straight line.

32. ABC, A′B′C′ are two triangles. Shew that if AA′, BB′, CC′ are concurrent and AB′, BC′, CA′ are concurrent, then also will AC′, BA′, CB′ be concurrent. (St. John's.)

33. If two complete quadrangles ABCD, A′B′C′D′ are such that five pairs of sides AB and A′B′, BC and B′C′, CA and C′A′, AD and A′D′, BD and B′D′ cut one another in five points on a straight line XY, then the remaining pair, CD and C′D′, will also meet on XY.

34. OX and OY are two fixed lines. SOS′ is a fixed line and S, S′ are fixed points on it. R is a variable point on a fixed line AB, SR and S′R meet OX and OY in P and P′. Prove that PP′ passes through a fixed point.

35. P, Q are any two points on the sides BC, CD respectively of a parallelogram ABCD. Prove that the lines joining B, C, D to the middle points of AP, PQ, QA respectively are concurrent. (Clare.)

36. HKF, EKG are drawn parallel to the sides AB, BC of a square ABCD, intersecting on the diagonal BD, H lying on AD and E on AB. Prove that HB, CK, DE are concurrent. (Jesus, etc.)

37. ABCD is a quadrilateral, E and F are any two points in AC and BD respectively; AF and BE meet in P, CF and DE meet in Q, and AD and CB meet in R. Prove that P, Q, R are collinear.

38. ABCD is a quadrilateral, the sides AB and DC being parallel, and E is the point of intersection of the diagonals, through which a line EO is drawn of any length parallel to AB; if OC and BD meet in H and OB meets CA in K, prove that KD and HA meet on OE. (St. Catharine's.)

39. A variable straight line turning about a fixed point U meets two fixed straight lines OX and OY in P and P′ respectively. If S and S′ are two fixed points collinear with O, SP and S′P′ meet in a point R whose locus is a straight line.

40. When three triangles are two by two in perspective, and have the same axis of perspective, their three centres of perspective are collinear.

41. When three triangles are two by two in perspective, and have the same centre of perspective, their three axes of perspective are concurrent.

CHAPTER II.

THE CONIC.

1. The properties of the "Conics" were first investigated by regarding them as the intersections, by planes, of a right circular cone. Hence we derive the name "Conic Sections."

It is proved in books dealing with the elementary geometry of these curves that all the species of conics can be obtained from a given right circular cone in this way. Viewing this result in the light of last chapter, we can deduce several properties of the conic directly from the corresponding properties of the circle, which we shall regard as proved in books dealing with the Modern Geometry of the Circle.

Let a right circular cone have O as vertex.

Let the plane π cut this cone in a circle.

Let the plane π' cut this cone in a conic.

Then, from the point of view of last chapter, we may regard the conic as the projection on to the plane π' of the circle lying in π.

We shall now give some immediate results:

I. **Every straight line cuts a circle in two points, real, coincident or imaginary.**(13)

Every straight line cuts a conic in two points, real, coincident or imaginary.

For the projection of a point of intersection of two curves is a point of intersection of the projected curves.

II. **From a given point can be drawn to a circle, two tangents, real, coincident or imaginary.**(14)

From a given point can be drawn to a conic, two tangents, real, coincident or imaginary.

For the projection of a tangent to a curve is a tangent to the projected curve.

III. **POLE AND POLAR. If X be a fixed point and XPQ a variable line cutting a circle in P and Q, and if Y be the harmonic conjugate of X with respect to P and Q, the locus of Y is a straight line x, called the "Polar Line" of X, and X is called the "Pole" of the line x.**

The Polar Line of X passes through the points of contact of the tangents from X to the circle.(15)

If X *be a fixed point and* XPQ *a variable line cutting a conic in* P *and* Q, *and if* Y *be the harmonic conjugate of* X *with respect to* P *and* Q, *the locus of* Y *is a straight line x, called the "Polar Line" of* X, *and* X *is called the "Pole" of the line x.*

The Polar Line of X *passes through the points of contact of the tangents from* X *to the conic.*

For a circle projects into a conic (Chap. II. Art. 1), a harmonic range into a harmonic range (Chap. I. Art. 7), and a straight line into a straight line (Chap. I. Art. 5).

IV. **CONJUGATE POINTS. Two points X and Y are said to be "conjugate" with respect to a circle when the line**

XY cuts the circle in two points P and Q harmonically separating X and Y.(16)

Two points X *and* Y *are said to be "conjugate" with respect to a conic when the line* XY *cuts the conic in two points* P *and* Q *harmonically separating* X *and* Y.

Note. **Two points conjugate with regard to a circle possess the property that the polar of either passes through the other.**(17)

Two points conjugate with regard to a conic possess the property that the polar of either passes through the other.

V. **CONJUGATE LINES. Two lines x and y are said to be "conjugate" with respect to a circle, when they harmonically separate the pair of tangents from the point xy to the circle.**(18)

Two lines x and y are said to be "conjugate" with respect to a conic when they harmonically separate the pair of tangents from the point xy to the conic.

Note. **Two lines conjugate with regard to a circle possess the property that the pole of either lies on the other.**(19)

Two lines conjugate with regard to a conic possess the property that the pole of either lies on the other.

VI. **CROSS-RATIO OF FOUR FIXED POINTS ON A CIRCLE. If A, B, C, D be four fixed points, and P a variable point lying on a circle, then the cross-ratio P[ABCD] is constant for all positions of P on the circle.**(20)

This is plain since

$$P[ABCD]=\frac{\sin APB \sin CPD}{\sin APD \sin CPB} \text{ (Chap. I. Art. 6),}$$

and since the angles subtended by two fixed points on the circumference at variable points on the circumference are equal or supplementary.

CROSS-RATIO OF FOUR FIXED POINTS ON A CONIC. If A, B, C, D be four fixed points, and P a variable point lying on a conic, then the cross-ratio P[ABCD] is constant for all positions of P on the conic.(21)

COROLLARY. *We see therefore that the following properties are projective* :

I. *Pole and Polar.*
II. *Conjugate Points.*
III. *Conjugate Lines.*

2. *The Projection of a Conic is a Conic.*

Let the given conic S lie in the plane π.

Let S′ be the projection in the plane π' of S lying in the plane π.

Since all straight lines lying in π meet the conic S in two points, all the projections of these straight lines will meet S′ in two points.

To be met by every straight line in *two* points is an exclusive property of the Conic Sections.

Hence S′ is a conic.

3. *Conjugate Imaginary Points.*

Let the lines X_1Y_1, X_2Y_2, X_3Y_3 be all real.

Let the conic S be also real.

Then, in the above figure, it is plain that
the line X_1Y_1 cuts the conic S in the *real* points A_1, B_1;
the line X_2Y_2 cuts the conic S in the *coincident* points A_2, B_2;

the line X_3Y_3 cuts the conic S in the *imaginary* points A_3, B_3;

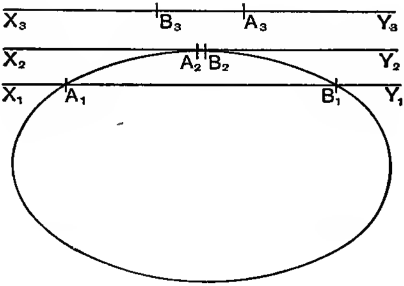

FIG. 15.

CONJUGATE IMAGINARY POINTS. The pair of imaginary points in which a real line cuts a real conic are called a pair of "conjugate imaginary points."(22)

[In Algebraic Geometry, the coordinates of one of a pair of conjugate imaginary points can be obtained from the coordinates of the other by changing $\sqrt{-1}$ into $-\sqrt{-1}$. Hence we obtain the name "conjugate imaginary."]

4. The following geometrical proposition will now be required:

If A, B and C, D be two given pairs of points lying on the same straight line, then, in general, one and only one pair of points X, Y can be found on the given line such that X, Y harmonically separate both A, B and C, D.(23)

To find XY, describe circles on AB and CD as diameters.

Construct the Limiting Points X and Y of the Coaxal System defined by the above two circles.

X and Y will be the points required.

For, by the theory of Coaxal Circles, A and B

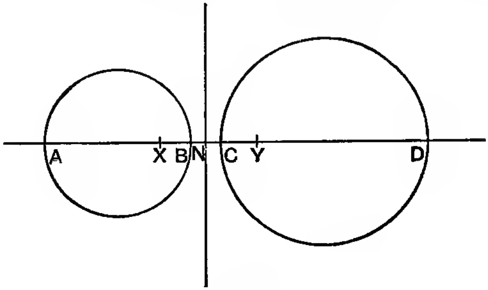

FIG. 16.

harmonically separate X and Y. Similarly with C and D.

The following considerations are important in what follows:

If the circles described on AB *and* CD *do not cut in real points as in the figure,* X *and* Y *will be real.*

If the circles described on AB *and* CD *cut in real points,* X *and* Y *will be imaginary.*

If O be a point from which emerge the two pairs of rays OA, OB and OC, OD, then, in general, one and only one pair of rays OX, OY can be found such that OX and OY harmonically separate both OA, OB and OC, OD.(24)

This last will be plain if we take a transversal cutting the four rays OA, OB, OC, OD in A, B, C, D respectively, and then find the two points X, Y harmonically separating both A, B and C, D as explained above.

5. The intersections of a given line and a circle harmonically separate all pairs of "conjugate points" on the given line.(25)

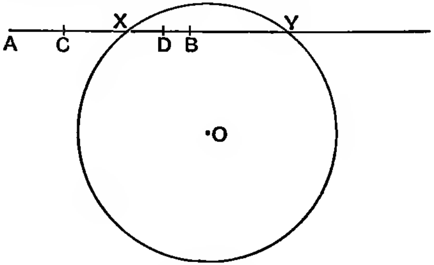

FIG. 17.

This is the fundamental property of "conjugate points" discussed in Chap. II. Art. 1, No. IV. In Fig. 17 the intersections X and Y harmonically separate the pairs of conjugate points A, B and C, D.

We may therefore regard the intersections of a given line and a circle as the pair of points harmonically separating any two pairs of conjugate points on the line.

6. *To find the Points in which a Given Circle cuts the Line at Infinity.*

Take A a point on the Line at Infinity, and let O be the centre of the circle.

Then the Polar Line of A will be the line OB through O perpendicular to OA.

Let OB meet the Line at Infinity in B.

Hence A and B will be a pair of Conjugate Points on the Line at Infinity.

In the same way, let **C** and **D** be another pair of Conjugate Points on the Line at Infinity.

Then, if **OX** and **OY** harmonically separate **OA**, **OB** and also **OC**, **OD** where **X** and **Y** are points on the Line at Infinity, **X** and **Y** will be the intersections of this circle with the Line at Infinity (Chap. II. Art. 5).

7. *All Circles cut the Line at Infinity in the same two Points.*

Let the two circles have as centres the points O_1 and O_2.

Then, if **A** and **B** be a pair of conjugate points on the Line at Infinity, with respect to the first circle, the angle AO_1B will be a right angle.

Now, since **A** is at infinity, the line O_2A is parallel to the line O_1A. Similarly, O_2B is parallel to O_1B. Hence, AO_2B is a right angle. Therefore, **A** and **B** are Conjugate Points with respect to the second circle.

Similarly, let **C** and **D** be Conjugate Points with respect to both circles.

Let now **X** and **Y** be the pair of points on the Line at Infinity that harmonically separate both **A**, **B** and **C**, **D**; therefore, **X** and **Y** will be the intersections of both circles with the Line at Infinity (Chap. II. Art. 6).

Thus, any *two* circles cut the Line at Infinity in the same two points, whence it is plain that *all circles cut the Line at Infinity in the same two points as any one circle.*

8. *The Circular Points at Infinity.*

The two fixed points situated on the Line at Infinity in any given plane, through which pass all circles lying in that plane, are called the Circular Points at Infinity in that plane, and are usually denoted by I and J.

We shall sometimes have occasion to denote the Circular Points at Infinity by other letters than I and J, in which case we shall append the suffix ∞, *e.g.* A_∞, B_∞.

9. *The Circular Points at Infinity are Conjugate Imaginary Points, as defined in Chap. II. Art. 3.*

For they are the imaginary intersections of a real circle with a real line, viz. the Line at Infinity (Chap. II. Art. 6).

10. Many things met with in Pure Geometry have to be regarded in a particular way before the methods of Projective Geometry can be applied to them. We proceed to investigate the principal of them.

11. *Projective Aspect of Perpendicular Lines.*

If OP and OQ be two perpendicular lines, then OP and OQ harmonically separate OI and OJ.

This is plain from Chap. II. Art. 6.

12. **ASYMPTOTES. The asymptotes of a given conic are the tangents at the points in which the curve cuts the Line at Infinity.**(26)

CENTRE OF A GIVEN CONIC. The asymptotes intersect in the centre of the conic, *i.e.* the centre is the pole of the Line at Infinity.(27)

13. *Projective Aspect of a Rectangular Hyperbola.*

Let the Rectangular Hyperbola cut the Line at

Infinity in H and K, and let O be the centre of the curve.

Then OH and OK are the asymptotes. But OH and OK are perpendicular by the definition of a Rectangular Hyperbola. Hence, OH and OK harmonically separate OI and OJ (Chap. II. Art. 6).

Thus I and J are harmonically conjugate with respect to H and K. Otherwise, I *and* J *are Conjugate Points with respect to the Rectangular Hyperbola.*

A Rectangular Hyperbola cuts the Line at Infinity in two points harmonically separating the Circular Points at Infinity.

14. *Projective Aspect of Concentric Circles.*

Let O be the common centre.

Then OI and OJ are the asymptotes of both circles, *i.e. all Concentric Circles touch one another at both* I *and* J.

15. *Projective Aspect of an Ellipse, Hyperbola, Parabola.*

A Real Conic which cuts the Line at Infinity in two imaginary points is called an Ellipse.(28)

A Real Conic which cuts the Line at Infinity in two real points is called a Hyperbola.(29)

A Real Conic which cuts the Line at Infinity in two coincident points is called a Parabola.(30)

16. *Projective Aspect of a Focus.*

The following is an elementary geometrical property of the conic :—

Let **PSQ** be a focal chord.

Let the tangents at **P** and **Q** intersect at **Z**. Then **Z** lies on the directrix, and **SZ** is perpendicular to **PQ**.

Now **Z** is the pole of the line **PQ**.

Here **SP** and **SZ** are Conjugate Lines, since the Pole of **SP** lies on **SZ**.

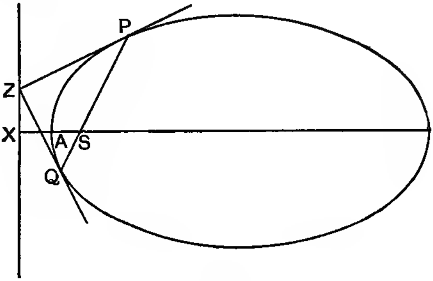

FIG. 18.

By Chap. II. Art. 1, the tangents from **S** to the conic harmonically separate all pairs of Conjugate Lines drawn from **S**. Hence, to find the tangents from **S** to the conic, we draw two pairs of Conjugate Lines from **S**, viz. SP_1, SZ_1 and SP_2, SZ_2, and then find the pair of lines that harmonically separate SP_1, SZ_1 and SP_2, SZ_2 by the method of Chap. II. Art. 4.

But when **S** is a focus, SP_1, SZ_1 and SP_2, SZ_2 are perpendicular pairs, and the pair of lines harmonically separating each of them is **SI**. **SJ**.

Hence the tangents from the Focus **S** *to the conic are* **SI** *and* **SJ**.

17. *The Four Foci of a Conic.*

Let **S** be a given conic and let **I**, **J** be the Circular Points at Infinity.

Let the two tangents from **I** intersect the two tangents from **J** in the four points S_1, S_2, S_3, S_4, which will therefore be the four foci of the conic.

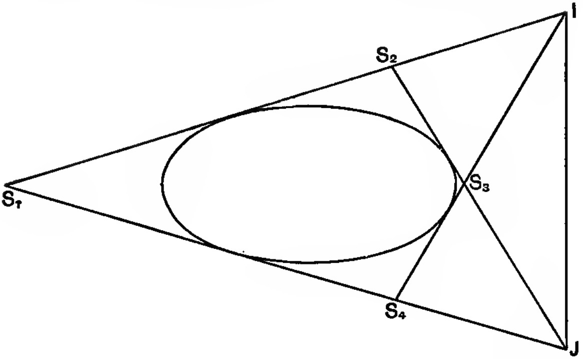

FIG. 19.

18. *Projective Aspect of two Equal Angles.*

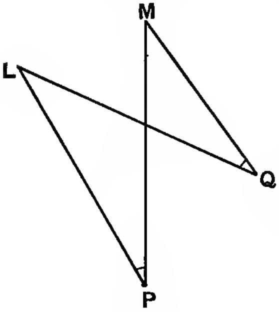

FIG. 20.

Let **LPM** and **LQM** be two equal angles.

Then the four points **LPQM** are concyclic.

Hence we may regard the points **P**, **Q**, **L**, **M** and the Circular Points **I** and **J** as all lying on the same conic.

$\therefore$ P[LIMJ] = Q[LIMJ] (Chap. II. Art. I.).

If the angles **LPM** *and* **LQM** *be equal, then* P[LIMJ] = Q[LIMJ].

N.B. It will be plain from a diagram which arms of the one angle correspond respectively to the arms of the other.

Conversely, if $P[LIMJ]=Q[LIMJ]$,

the angle $LPM =$ *the angle* LQM.

(We know that if A, B, C, D be four fixed points and P a moving point such that $P[ABCD]$ is constant, then P describes a conic passing through A, B, C, D.)

Hence, since $P[LIMJ]=Q[LIMJ]$,

$\therefore$ the points L, P, M, Q, I, J lie on a conic (Chap. II. Art 1).

i.e. L, P, M, Q are concyclic (since the conic passes through the Circular Points at Infinity).

Hence the angles LPM and LQM are equal.

COROLLARY. *To find the projective meaning of two conics having equal eccentricity, i.e. "similar conics."*

Let the centres of the two conics (*e.g.* hyperbolas) be C_1 and C_2.

Let the first conic cut the Line at Infinity in H_1 and K_1.

Let the second conic cut the Line at Infinity in H_2 and K_2.

Now, if two conics have equal eccentricity, the angles between their asymptotes are equal (since the eccentricity $= \sec \alpha$, where $2\alpha =$ angle between asymptotes).

Hence the angle $H_1C_1K_1 =$ the angle $H_2C_2K_2$.

i.e. $C_1[H_1IK_1J]=C_2[H_2IK_2J]$.

Hence two conics of equal eccentricity cut the Line at Infinity in pairs of points forming with I *and* J *ranges of equal cross-ratio.*

N.B. **Similar and similarly situated conics cut the Line at Infinity in the same two points.**(31)

19. *To project a Given Conic into a Circle and at the same time any given Line to Infinity.*

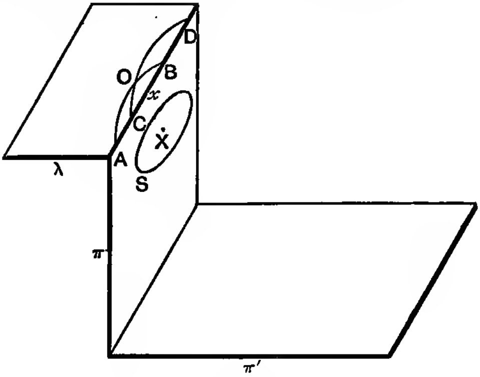

FIG. 21.

Let x be the line (lying in the plane π) which we wish to project to infinity.

Let **X** be the pole of x with respect to the conic **S**.

Let **A**, **B** and **C**, **D** be any two pairs of conjugate points with respect to **S**, lying on x.

Then the intersections of x with **S** may be regarded as the pair of points harmonically separating **A**, **B** and **C**, **D** (Chap. II. Art. 5).

Take any plane λ through x (other than π).

Let π' be a plane parallel to λ.

Let the circles in the plane λ on **AB** and **CD** as diameters cut in **O**.

Now, by Chap. II. Art. 4, the limiting points of the coaxal system defined by these two circles harmonically separate **A**, **B** and **C**, **D**, *i.e.* they are the intersections of x and **S**, as explained above.

Hence, if the intersections of x and **S** *be imaginary, the point* **O** *will be real* (Chap. II. Art. 4).

Now project x and **S** on to π', with **O** as vertex of projection.

S projects into the conic **S**$'$ (Chap. II. Art. 1).

x', the projection of x, lies at infinity ($\because$ the plane **O**x is parallel to π').

X$'$ is the polar of x' with respect to **S**$'$, *i.e.* **X**$'$ is the centre of **S**$'$ (Chap. II. Art. 12).

The line **OA** is parallel to the line **X**$'$**A**$'$.

(For since the plane λ is parallel to the plane π', any plane must cut λ and π' in lines parallel to one another. Now the plane **OAX** cuts λ in **OA** and π' in **X**$'$**A**$'$. Hence **OA** and **X**$'$**A**$'$ are parallel.)

Similarly **OB** is parallel to **X**$'$**B**$'$;

OC is parallel to **X**$'$**C**$'$;

OD is parallel to **X**$'$**D**$'$.

Now, since **AOB** is a semi-circle,

$\therefore$ **OA** and **OB** are perpendicular.

Similarly **OC** and **OD** are perpendicular.

Hence **X**$'$**A**$'$ and **X**$'$**B**$'$ are perpendicular

and **X**$'$**C**$'$ and **X**$'$**D**$'$ are perpendicular.

But **A** and **B** are conjugate points with respect to **S**; also **C** and **D** are conjugate points with respect to **S**.

Hence **A**$'$ and **B**$'$ are conjugate points with respect to **S**$'$; also **C**$'$ and **D**$'$ are conjugate points with respect to **S**$'$.

Now, since **X**$'$**A**$'$, **X**$'$**B**$'$ and **X**$'$**C**$'$, **X**$'$**D**$'$ are each perpendicular pairs of lines,

$\therefore$ **I**$'$, **J**$'$ harmonically separate **A**$'$, **B**$'$ and **C**$'$, **D**$'$ (Chap. II. Art. 11).

But the intersections of **S**$'$ and the Line at Infinity on π' harmonically separate **A**$'$, **B**$'$ and **C**$'$, **D**$'$, since each of these is a pair of conjugate points with respect to **S**$'$.

Hence the points of intersection of **S**$'$ and the Line at Infinity on π' are **I**$'$ and **J**$'$.

$\therefore$ **S**$'$ is a circle.

If a real line x cut a real conic **S** *in a pair of imaginary points, we can by a real projection from a real point* **O** *project* **S** *into a circle, and at the same time x into the Line at Infinity.*

The following is a more convenient form of the above proposition and of wider application:

We can always, by a real projection, project any two given conjugate imaginary points into the Circular Points at Infinity on π'.

We therefore select those two points on the given figure which we think will simplify the figure most, and project them into the Circular Points at Infinity.

Observation I. *The Line at Infinity on π projects into a line in the finite part of the plane π'.*

Observation II. *The Circular Points at Infinity on π project into a pair of points (imaginary if the vertex of projection be real) lying in the finite part of π'.*

20. *Projections where the Vertex of Projection* **O** *is an Imaginary Point.*

If in last article the line x cut the conic **S** in real points, the Limiting Points of the Coaxal system of circles defined by the circles on **AB** and **CD** as diameters are real, and hence the circles on **AB** and

CD as diameters intersect in imaginary points. O will in that case be imaginary, and hence real points, etc., on the plane π will not project into real points, etc., on the plane π'.

Next article shews the method of reasoning to be adopted when the two points, which we wish to project into the Circular Points at Infinity, are real.

21. *Method of Reasoning when the two Points to be projected into the Circular Points at Infinity are real.*

An example will make the method of procedure plain.

Consider the case of two circles cutting in the real points H, K. Let P be a point on the line HK, and let PQ and PR be the tangents to the two circles respectively. Then $PQ^2 = PH . PK = PR^2$.

Hence $PQ = PR$.

Thus, in the case when two circles cut in real points, it can be easily proved that the locus of points such that the tangents from them to each of the two circles are equal, is a straight line.

This proposition having been now established, it *must* be quite possible to prove it by the methods of Analytical Geometry.

But the methods of Analytical Geometry would in general take no cognizance of whether the points of intersection were real or imaginary.

Hence the chain of equations that prove the proposition in the case when H and K are real, prove the proposition at the same time when H and K are imaginary.

This method of reasoning is called the Principle of Continuity in Geometry.

Hence a proposition is first proved by a Real Projection for that case of the figure when the two points to be projected into the Circular Points at Infinity are conjugate imaginary points, and then the proposition is extended to the case of real points by the Principle of Continuity.

22. It will probably be found best, as before, to draw the projected figure directly from the enunciation.

23. Example 1.

How many conics can be drawn through five given points?

Project two of the points into the Circular Points at Infinity, and then it becomes a question of how many circles can be drawn through three given points. Obviously *one.*

Example 2.

How many conics can be drawn through two given points to touch three given lines?

Project the two given points into the Circular Points at Infinity, and then we have to find how many circles can be drawn to touch three given lines. Obviously *four,* viz. the inscribed and the three escribed circles.

Example 3.

If two conics touch at O, *and if a line through* O *meet the conics again in* P *and* Q *respectively, the tangents at* P *and* Q *to the two conics will intersect in a point* T *whose locus is a straight line.*

Two conics intersect in four points, and in the above case two of these points are coincident at O. Let the remaining two points be A and B, and project these into the Circular Points at Infinity A'_∞, B'_∞.

Let the tangents at P′ and Q′ meet the tangent at O′ in L′ and M′ respectively.

Then $L'O' = L'P'$ (tangents to the same circle).

$$\therefore \angle L'O'P' = \angle L'P'O'.$$

So $\angle M'O'Q' = M'Q'O'$.

$$\therefore \angle L'P'O' = \angle M'Q'O',$$

$$\therefore L'P' \text{ is parallel to } M'Q',$$

i.e. the point of intersection T′ of L′P′ and M′Q′ lies on the Line at Infinity $A'_\infty\ B'_\infty$.

Hence the locus of T is the line AB.

Example 4.

In the above Example 3, to find the locus of R *the harmonic conjugate of* O *with respect to* P *and* Q.

Project as above.

Let the diameters of the two circles be O′L′ and O′M′. Draw R′N′ perpendicular to O′Q′, meeting O′L′M′ in N′.

Since O′P′L′ and O′Q′M′ are semi-circles,

$\therefore \angle O'P'L'$ and $\angle O'Q'M'$ are each right angles.

Hence P′L′, R′N′, Q′M′ are all parallel.

Since [O′P′R′Q′] is harmonic,

$\therefore$ [O′L′N′M′] is harmonic.

N′ is therefore a fixed point, and the locus of R′ is a circle on O′N′ as diameter.

Hence the locus of R is a conic passing through A and B and touching each of the original conics at O.

Example 5.

If A, B, C, D *be any four fixed points in a plane,* *and if* P *be a moving point such that the lines* PA, PB *harmonically separate* PC, PD, *then* P *describes a conic passing through* A, B, C, D *and having* AB *and* CD *as conjugate lines.*

Project A, B into the Circular Points at Infinity A'_∞, B'_∞.

Then P′C′, P′D′ harmonically separate $P'A'_\infty$, $P'B'_\infty$, *i.e.* P′C′, P′D′ are perpendicular.

Hence the locus of P′ is the semi-circle on C′, D′.

Also $A'_\infty B'_\infty$ and C′D′ are Conjugate Lines, since the centre of the circle, *i.e.* the pole of $A'_\infty B'_\infty$, lies on C′D′.

Example 6.

If A *and* B *be two fixed points on a conic, and* P *a moving point such that the tangents from* P *to the conic form with* PA *and* PB *a pencil of constant cross-ratio, then* P *will describe a conic touching the original conic at* A *and* B.

Project A and B into the Circular Points at Infinity A'_∞ and B'_∞.

We have then to prove that if the tangents from P′ to the circle contain a constant angle, the locus of P′ will be a concentric circle (Chap. II. Art. 14), which is obvious.

Example 7.

Given a conic S, *and two points* A *and* B *not upon it, we can project this conic into a conic* S′ *having* A′ *and* B′ *as foci.*

Let a tangent from A meet a tangent from B in H, and the other tangent from A meet the other tangent from B in K.

Project H and K into the Circular Points at Infinity.

Example 8.

To project a given conic into a circle, and at the same time an inscribed triangle into an equilateral triangle.

Let the inscribed triangle be ABC.

Let the tangent at A meet BC in L.
” ” ” B ” CA ” M.
” ” ” C ” AB ” N.

Project the points in which LM meets the Conic into the Circular Points at Infinity.

Because in the circle $A'B'C'$ the tangent at A' is parallel to $B'C'$, ∴ $A'B'=A'C'$.

Similarly $B'C'=B'A'$.

Hence $A'B'C'$ is an equilateral triangle.

N.B. This affords a proof that L', M', N' *are collinear, since the tangent at* C' *is parallel to* $A'B'$.

Example 9.

S *is a given conic and* ABC *a given triangle. If the polar line of* A *meet* BC *in* L,
” ” B ” CA ” M,
” ” C ” AB ” N,
to prove that L, M, N *are collinear.*

Project the points, in which LM meet S, into the Circular Points at Infinity.

Let O' be the centre of the circle thus obtained.

Since the Polar Line of A' is perpendicular to $O'A'$ by a known theorem in Modern Geometry, and parallel to $B'C'$ by hypothesis, ∴ $O'A'$ is perpendicular to $B'C'$. So $O'B'$ is perpendicular to $C'A'$. Hence O' is the orthocentre of the triangle $A'B'C'$. Thus $O'C'$ is perpendicular to $A'B'$, whence the Polar Line of C' is parallel to A′B′. Consequently N′ lies on the Line at Infinity L′M′.

∴ L, M, N are collinear.

Important Type of Problem.

Angles clustering round a point.

It frequently happens that we are required to prove theorems about angles that cluster round a point.

Thus, let P be the point and H_∞, K_∞ the Circular Points at Infinity.

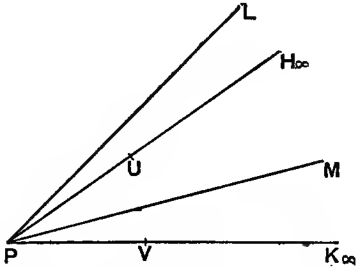

FIG. 22.

Let LPM be an angle.

Let U and V be two points on PH_∞ and PK_∞ respectively.

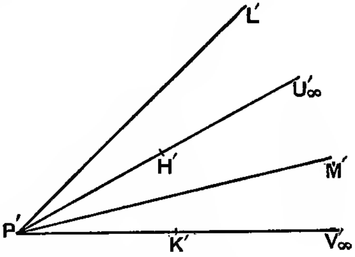

FIG. 23.

Project U and V into U'_∞ and V'_∞.

Then, since $P[LH_\infty MK_\infty]=P'[L'U'_\infty M'V'_\infty]$, *the angles round* P *are unaltered in magnitude after projection,* by Chap. II. Art. 18.

A few examples will illustrate cases which can be successfully attacked by the above method.

Example 10.

If O *be a fixed point on a conic* S *and* OP, OQ *two perpendicular chords, to prove that* PQ *passes through a fixed point.*

Let $A_\infty B_\infty$ be the Circular Points at Infinity.

Let OA_∞, OB_∞ cut the conic in H, K respectively.

Project H, K into H'_∞, K'_∞, the Circular Points at Infinity.

A', B' will now be two points in the finite part of the plane π'.

Since $O'P'$, $O'Q'$ harmonically separate $O'H'_\infty$, $O'K'_\infty$, $\therefore$ $O'P'$, $O'Q'$ are perpendicular.

Now it is plain that in a circle $P'Q'$ passes always through the centre (*i.e.* the pole of $H'_\infty K'_\infty$), since $P'O'Q'$ is a right angle.

Hence PQ passes always through the pole of HK (called the Frégier Point of O), which lies on the normal at O.

Example 11.

If a chord of a conic subtend a constant angle at a focus, and a point on the chord produced be taken, such that the intercept on the chord between this point and the conic subtends an equal angle at the same focus, the locus of the point is a conic having the same focus and directrix.

Let the focus be S, the chord PQ, and the moving point R.

Let the lines joining S to the Circular Points at Infinity touch the conic at H and K respectively.

Project H and K into H'_∞ and K'_∞.

The angles round S are unaltered in magnitude by this projection.

Also S′ is the centre of the circle thus obtained.

Let P′Q′ be the variable chord subtending a fixed angle at S′, and let P′S′R′ = the angle P′S′Q′.

Plainly R′ describes a circle concentric with the above circle, *i.e.* touching $S'H'_\infty$ and $S'K'_\infty$ at H'_∞ and K'_∞.

Hence in the original figure the locus of R is a conic touching SH and SK at H and K, *i.e.* the locus of R is a conic having the same focus and directrix as the given conic.

24. *Generalisation by Projection.*

This is the reverse process to the above.

Given a geometrical theorem involving metrical properties, *e.g.* magnitudes of angles, circles, ratios of collinear segments of lines, etc., it is often possible by projecting from an arbitrary vertex of projection on to another plane to derive a still more general proposition about cross-ratios of pencils, conics, cross-ratios of points on a line, etc.

In virtue of using an arbitrary vertex of projection, the Circular Points at Infinity in π become points in the finite part of π'.

Hence to generalise by projection, draw the figure giving to the metrical properties (where possible) their projective interpretations as explained in Chap. II.

Arts. 6 to 20, and then regard the Circular Points at Infinity as ordinary points.

A few examples will make this plain.

25. Example 1.

The line joining the centre O *of a circle to the point of contact* P *of a tangent is perpendicular to that tangent.*

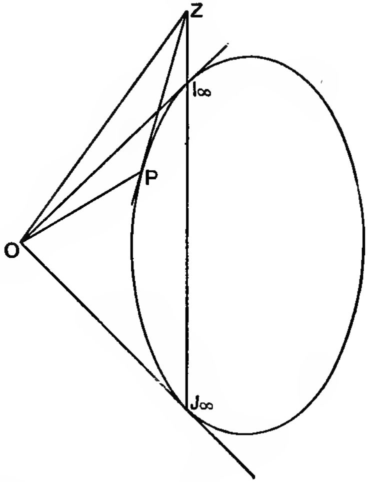

FIG. 24.

Generalisation.

If O *be a point whose Polar Line meets a conic in* I *and* J, *and if* P *be a point on the conic, the tangent at which meets* IJ *in* Z, *then will* OP *and* OZ *harmonically separate* OI *and* OJ.

Example 2.

If two circles be concentric, and a chord of one touch the other, it is bisected at the point of contact.

Generalisation.

If two conics touch at I *and* J, *and if a chord* PQ *of one touch the other at* L *and cut* IJ *in* M, *then* L, M *harmonically separate* P, Q.

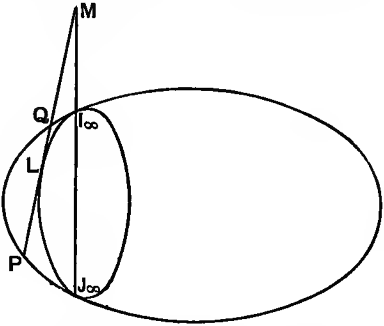

FIG. 25.

Example 3.

If A *be a fixed point on a circle, and if* PQ *be a chord subtending a constant angle at* A, *then* PQ *will envelope a circle concentric with the given circle.*

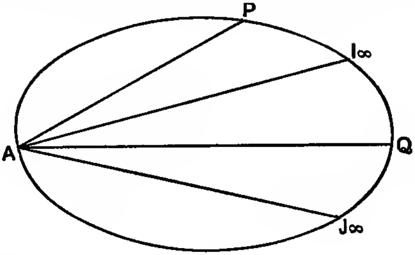

FIG. 26.

Generalisation.

If A *be a fixed point on a conic,* I *and* J *two other fixed points, and if* AP, AQ *be two variable chords such* *that* A[PIQJ] *is constant, then will* PQ *envelope a conic touching the original conic at* I *and* J.

Example 4.

If P *be a point on the directrix to a parabola, and if* PQ, PR *be the tangents from* P *to the parabola, then* PQ, PR *will be perpendicular.*

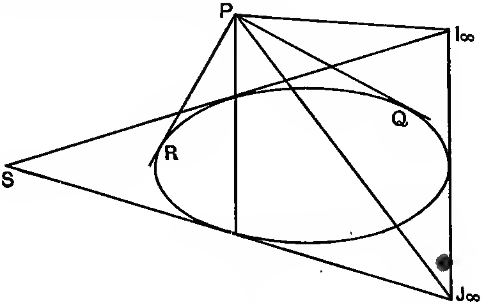

FIG. 27.

Generalisation.

If a conic be inscribed to the triangle SIJ, *and if* P *be a point on the Polar Line of* S, *then the tangents from* P *to the conic harmonically separate* PI *and* PJ.

26. It will not be found most convenient in all cases to project the given figure so that the conic involved becomes a circle. Sometimes projection into a parabola or hyperbola gives a more convenient solution.

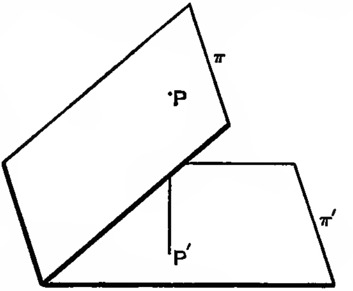

FIG. 28.

27. *Orthogonal Projection.*

Let P be a point on the plane π in Fig. 28.

Drop a perpendicular **PP′** from **P** on to π'.

P′ is called the orthogonal projection of **P** on the plane π' (π and π' being always supposed not perpendicular).

28. *The orthogonal projection of a straight line is a straight line.*

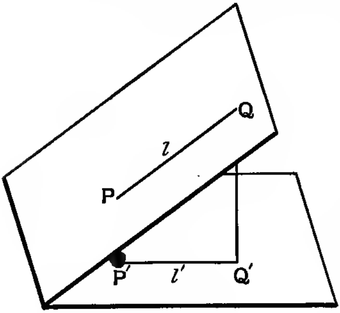

FIG. 29.

Let l be a straight line lying in π.

Drop the perpendicular **PP′**.

Draw the plane λ containing l and **PP′**, and let it cut the plane π' in the line l'.

Draw in λ the line **QQ′** perpendicular to l'.

Since the plane λ is drawn containing the normal **PP′** to π', $\therefore$ λ is perpendicular to π'.

Since **QQ′** lies in λ and is perpendicular to the common section l' of λ and π', $\therefore$ **QQ′** is perpendicular to π'.

Similarly we can shew that the locus of the feet of all perpendiculars drawn from points on l on to π' is l'.

Hence the straight line l' is the orthogonal projection of the straight line l.

29. *The orthogonal projection of two parallel lines is two parallel lines.*

For let l_1 and l_2 be the two parallel lines in π.

Let l'_1 and l'_2 be their orthogonal projections in the plane π'.

Now the projection of the intersection of l_1 and l_2 is the intersection of l'_1 and l'_2.

But l_1 and l_2 meet at infinity.

Hence l'_1 and l'_2 intersect at infinity,

i.e. l'_1 and l'_2 are parallel.

30. *If two lines be parallel, the ratio of their lengths is unaltered by orthogonal projection.*

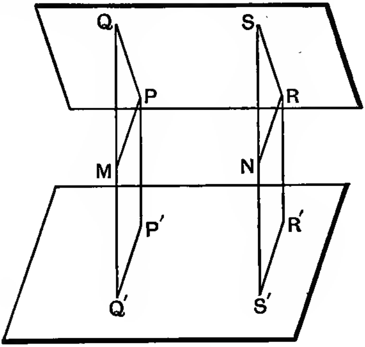

Fig. 30.

Draw PM and RN parallel to P'Q' and R'S' respectively.

Then PQ is parallel to RS,

PM „ „ RN,

QM „ „ SN.

Hence the triangles PQM and RSN are similar;

$$\therefore \frac{PQ}{RS} = \frac{PM}{RN}$$

$$= \frac{P'Q'}{R'S'}.$$

COROLLARY I. *A bisected line projects into a bisected line, in orthogonal projection.*

COROLLARY II. *Cross-ratios are unaltered by orthogonal projection.*

31. *An ellipse can be projected orthogonally into a circle.*

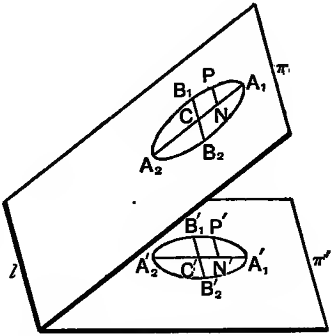

FIG. 31.

Choose the plane π' so that the line of intersection of π' and π is parallel to the minor axis B_1B_2.

Let the angle between π and π' be θ where

$$\cos\theta = \frac{B_1C}{A_1C}.$$

Then it is plain that

$$C'A'_1 = CA_1\cos\theta = B_1C = B'_1C'. \quad \text{............}(1)$$

But by the properties of an ellipse,

$$\frac{PN^2}{A_1N \cdot A_2N} = \frac{B_1C^2}{A_1C^2}. \quad \text{..................}(2)$$

Also

$$\left.\begin{aligned} P'N' &= PN \\ B'_1C' &= B_1C \\ A'_1C' &= A_1C\cos\theta \\ A'_1N' &= A_1N\cos\theta \\ A'_2N' &= A_2N\cos\theta \end{aligned}\right\} \text{..................}(3)$$

Hence, by (1), (2) and (3),

$$\frac{P'N'^2}{A_1'N' \cdot A_2'N'} = \frac{B_1'C'^2}{A_1'C'^2} = 1\,;$$

$$\therefore\ P'N'^2 = A_1'N' \cdot A_2'N'.$$

Hence the locus of P′ is a circle.

COROLLARY. *Conjugate diameters of the Ellipse project into perpendicular diameters of the Circle.* For the tangent at the extremity of either diameter is parallel to the other diameter.

32. *A hyperbola can be orthogonally projected into a Rectangular Hyperbola.*

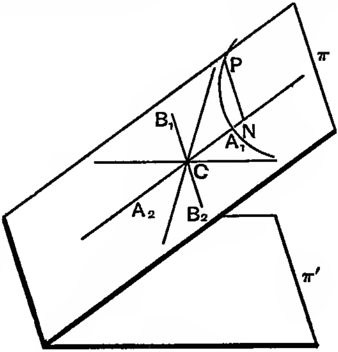

FIG. 32.

Let, as in last article, the angle θ between π' and π be so chosen that $\cos\theta = \frac{B_1C}{A_1C}$.

Then it can be proved exactly, as in last article, that the given hyperbola projects into a hyperbola, in which the major axis is equal to the minor axis. The hyperbola in π' will therefore be rectangular.

33. *The Line at Infinity in π projects orthogonally into the Line at Infinity in π'.*

For a point at infinity on π obviously projects orthogonally into a point at infinity on π', since points at a finite distance on π project orthogonally into points at a finite distance on π', and points at a finite distance on π' correspond to points at a finite distance on π. Here a point at infinity on π cannot project orthogonally into a point at a finite distance on π'.

34. *The asymptotes of a conic* S *project orthogonally into the asymptotes of the conic* S′.

For if the Line at Infinity in π cut S in H and K, then the Line at Infinity in π' cuts S′ in H′ and K′, by Chap. II. Art. 33.

Also, the tangents at H and K project into the tangents at H′ and K′ respectively.

35. *The orthogonal projection of an ellipse, hyperbola, parabola is an ellipse, hyperbola, parabola.*

For the points at infinity on the conic S project into the points at infinity on the conic S′.

If imaginary, real or coincident before projection, their character is unaltered thereafter.

36. *Any two lines can be projected orthogonally so as to contain a right angle.*

Let OP, OQ be the given lines, and let OH, OK be the bisectors of the angles between them.

Take π' so that the line $\pi\pi'$ is parallel to OK.

Let the angle θ between π and π' be such that

$$\cos\theta = \frac{\mathsf{LM}}{\mathsf{OM}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where LM is the perpendicular from any point L on OQ to OH.

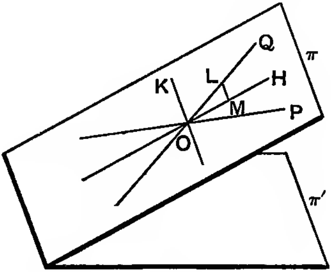

FIG. 33.

Then $$\mathsf{L'M'} = \mathsf{LM} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and $$\begin{aligned}\mathsf{O'M'} &= \mathsf{OM}\cos\theta \\ &= \mathsf{LM} \text{ by } (1) \\ &= \mathsf{L'M'} \text{ by } (2).\end{aligned}$$

Hence L′O′M′ is half a right angle.
Thus P′O′Q′ = right angle.

37. Example 1.

If an ellipse be projected orthogonally into a circle, the centre of the ellipse projects into the centre of the circle.

For all chords through the centre O of the ellipse are bisected by the curve, hence all chords through O′ are bisected by the circle which is the projection of the ellipse (Chap. II. Art. 30, Cor. I.). Hence O′ must be the centre of the projected curve.

Example 2.

In an ellipse the locus of the middle points of a system of parallel chords is a straight line.

Project the ellipse orthogonally into a circle, in which case the proposition is evident, if we remember that the mid-point of a line projects orthogonally into the mid-point of the projected line.

Example 3.

In an ellipse, the ratio of the rectangles contained by the segments of any two chords is equal to the ratio of the rectangles contained by the segments of any two chords parallel to them respectively.

Let CL and CM be the semi-diameters parallel to the chords POQ and ROS respectively.

Project the ellipse orthogonally into a circle.

Then, since ratios of parallel lines are unaltered by orthogonal projection,

$$\therefore \frac{\mathrm{OP}.\mathrm{OQ}}{\mathrm{CL}^2}=\frac{\mathrm{O'P'}.\mathrm{O'Q'}}{\mathrm{C'L'}^2}\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$\frac{\mathrm{OR}.\mathrm{OS}}{\mathrm{CM}^2}=\frac{\mathrm{O'R'}.\mathrm{O'S'}}{\mathrm{C'M'}^2}. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

But by properties of the circle

$$\mathrm{O'P'}.\mathrm{O'Q'}=\mathrm{O'R'}.\mathrm{O'S'}\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and

$$\mathrm{C'L'}^2=\mathrm{C'M'}^2.$$

∴ from (1), (2) and (3),

$$\frac{\mathrm{OP}.\mathrm{OQ}}{\mathrm{CL}^2}=\frac{\mathrm{OR}.\mathrm{OS}}{\mathrm{CM}^2},$$

$$i.e.\ \frac{\mathrm{OP}.\mathrm{OQ}}{\mathrm{OR}.\mathrm{OS}}=\frac{\mathrm{CL}^2}{\mathrm{CM}^2},$$

whence the proposition is evident.

Example 4.

In an ellipse, if CT *meet the Polar Line of* T *in* V *and the curve in* P, $CV \cdot CT = CP^2$.

Project the ellipse orthogonally into a circle.

We have to prove $\dfrac{CV \cdot CT}{CP^2} = 1$,

$$i.e.\ \frac{C'V' \cdot C'T'}{C'P'^2} = 1 \text{ (Chap. II. Art. 30)}$$

(since CV, CP, CT are all segments lying in the same straight line), which is evident for a circle.

Example 5.

To project orthogonally a parabola into another parabola, with the projection of a given point on the curve as vertex.

Let P be a point on the parabola, PT the tangent at P and PH the diameter through P. Let H be the point of contact of the parabola with the Line at Infinity.

Project the curve orthogonally, so that $P'T'$ and $P'H'$ are perpendicular. By Chap. II. Art. 35, H' is the point of contact of the parabola with the Line at Infinity, whence the proposition is evident.

Example 6.

Prove that in the case of a hyperbola, the parts of a straight line intercepted between the asymptotes and the curve are equal.

Let C be the centre of the hyperbola, CH and CK the asymptotes.

Let the chord P_1P_2 cut the asymptotes in Q_1 and Q_2 respectively.

Let V be the middle point of P_1P_2.

Project orthogonally, so that $C'V'$ and $P_1'P_2'$ become perpendicular.

We then obtain a hyperbola having $C'V'$ as major axis and $C'H'$, $C'K'$ as asymptotes.

From symmetry $\dfrac{P_1'Q_1'}{P_2'Q_2'}=1$,

and since the ratios of segments of a chord are unaltered, we obtain that

$$\frac{P_1Q_1}{P_2Q_2}=1$$

38. The following is an immediate corollary from Chap. II. Art. 31, and is worthy of note.

We can, by a real orthogonal projection, project any two conjugate imaginary points on the Line at Infinity in π *into the Circular Points at Infinity on* π', *if* π' *be suitably chosen.*

This theorem is often useful in solving problems. When the given points in π are not conjugate imaginary points, we use the Principle of Continuity as explained in Chap. II. Art. 21.

EXAMPLES II.

1. Two conics touch at O and meet again in A and B. A common tangent touches them at P and Q. Shew that OP and OQ are harmonically conjugate to OA and OB.

2. Two conics touch at O and meet again in A and B. A common tangent touches them at P and Q respectively. The tangent at O meets PQ in R and AB meets PQ in S. Shew that R and S are harmonically conjugate to P and Q.

3. If two conics touch at O and meet again in A and B, the meeting point of the two common tangents, the poles of AB with respect to the two conics and the point of contact O are collinear.

4. If two conics touch at O and meet again in A and B; if also the points of contact of the two common tangents be P_1 and P_2 on the first conic, and Q_1 and Q_2 on the second, prove that P_1P_2, Q_1Q_2, AB and the tangent at O are concurrent and form a harmonic pencil.

5. Let the point of concurrency in question (4) be U and the meeting point of the two common tangents be T; then TU and TO are harmonically conjugate to the two common tangents.

6. Shew that two conics touching at a point can be projected into two confocal parabolas.

7. Two conics S_1 and S_2 cut in the points K, L, M, N. The tangents at K and L to S_1 meet in U_1, and the tangents to S_2 at the same points meet in U_2. Similarly the tangents to the conics at M and N meet in V_1 and V_2. Prove that $U_1U_2V_1V_2$ are collinear.

8. Two conics S_1 and S_2 touch at O and meet again in E and F. A_1 is the pole of EF to S_1 and A_2 to S_2. OP_1P_2

meets S_1 in P_1 and S_2 in P_2. Shew that A_1P_1 and A_2P_2 meet on EF.

9. Two conics have double contact at A and B. P is a variable point on the first. Prove that the cross-ratio of the pencil formed by the pair of tangents from P to the second and the two lines PA and PB is constant.

10. If two conics have double contact, prove that the polars of points on the first with respect to the second envelope a conic having double contact with the first two.

11. If three conics all pass through the same two points, their three other chords of intersection are concurrent.

12. Two conics meet in A_1 and A_2. A_1P_1 is a chord meeting the first in P_1 and the second in Q_1. $A_2P_2Q_2$ is a similar chord. Shew that P_1P_2 and Q_1Q_2 meet on the other chord of intersection of the two conics.

13. If two conics have double contact, the polars of any point whatsoever with respect to the two conics meet on the chord of contact.

14. If two conics touch at A and B and the tangents at A and B meet in O, then the line joining the poles with regard to both conics of any given straight line passes through O.

15. ABC is a triangle inscribed in a conic. The tangents at A, B, C meet the opposite sides BC, CA, AB respectively in D, E, F. We can prove by joining DE and projecting into the circular points at infinity the points where DE meets the conic that D, E, F are collinear. Shew that if the tangents from D, E, F to the conic (other than DA, EB, FC) touch at the points P, Q, R respectively, AP, BQ, CR are concurrent.

16. If a quadrilateral circumscribe a conic, shew that its three diagonals form a self-conjugate triangle.

17. Two conics have double contact and are such that *one* triangle can be inscribed in the one and circumscribed to the other. Shew that an infinite number of triangles can be so inscribed and circumscribed.

18. Two conics have double contact and are such that triangles can be inscribed in the one and circumscribed to the other. If ABC be one of these triangles, and have its vertices lying on the circumscribed conic, and if D, E, F be where BC, CA, AB touch the inscribed conic respectively, prove that the point in which AD, BE, CF meet is constant for all positions of the triangle.

19. Also shew that, in question 18, the tangents at A, B, C meet the opposite sides in three points which lie on a fixed straight line for all positions of the triangle.

20. ABC is a triangle having a conic inscribed in it and touching its sides at the points D, E, F respectively. AD, BE, CF meet the conic again in the points P, Q, R respectively. Shew that the tangents at P, Q, R meet BC, CA, AB respectively in three collinear points.

21. Two conics touch at O and cut again in A and B. Shew that all chords through O are cut by the conics, by AB, and by O in four points that have a constant cross-ratio.

22. Two conics have double contact, A and B being the points at which they touch. If the tangents at A and B meet in O, prove that all lines through O cut the two conics in four points having a constant cross-ratio.

23. Two conics meet in the four points A, B, C, D. A chord PAQ is drawn through A meeting the conics in P and Q. Shew that the pencil B[PCQD] has a constant cross-ratio.

24. A series of conics pass through two fixed points A and B, and touch two fixed lines OX and OY. Prove that the chords of contact all pass through one or other of two fixed points.

25. A series of conics pass through two fixed points A, B and touch two fixed lines OX, OY. Prove that the locus of the pole of AB is one or other of two fixed lines passing through O and forming a harmonic pencil with OX and OY.

26. Given a conic, a point, and a line, shew that we can project the line to infinity and at the same time the point into a focus.

27. O is a given point and PQ is a given chord of a conic cutting the polar of O in U. V is the harmonic conjugate of U with respect to P and Q. If the polar of O meet the conic in A and B, shew that VA and VB are harmonically conjugate to VU and VO.

28. O is a fixed point and l is a fixed line. Prove that the locus of the harmonic conjugates of all points P on l, with respect to the pair of points in which OP meets a conic, is another conic passing through O and the points in which l meets the given conic.

29. ABC is a triangle. X and Y are two given points. A conic passes through the four points ABXY and has AB and XY as conjugate lines. Another conic passes through ACXY and has AC and XY as conjugate lines. Prove that these two conics intersect on BC.

30. Prove that, if through the extremities of two conjugate semi-diameters CP and CD of an ellipse parallel chords PQ and DE are drawn, then CQ and CE are conjugate. (Trinity, etc.)

31. A, B are two fixed points, and a variable circle through them cuts a fixed circle in C, D. Prove that the line joining the intersections of AC, BD and AD, BC passes through a fixed point. (Peterhouse, etc.)

32. ABCD is a quadrilateral inscribed in an ellipse. AB, DC meet in P, AC, BD in Q; shew that the poles of BC and AD lie on PQ. (Peterhouse, etc)

33. A and B are two points on a conic. O is a point on AB. Two chords P_1OP_2 and Q_1OQ_2 meet in O. Shew that if R be a point on the conic $R[P_2AQ_2B] = R[P_1BQ_1A]$.

34. A, B, P are three points on a conic and O is the pole of AB. OP meets AB in X, and the tangent at P meets AB in T. Prove that [AXBT] is harmonic.

35. P_1 and P_2 are two points, the tangents from which to a conic are P_1Q_1, P_1R_1 and P_2Q_2, P_2R_2, and touch at Q_1, R_1, Q_2, R_2. P_1P_2 meets Q_1R_1 and Q_2R_2 in U_1 and U_2 respectively, and Q_1R_1, Q_2R_2 meet in V. Prove that P_1P_2 is the polar of V and that $[U_1Q_1VR_1]$ and $[U_2Q_2VR_2]$ are each harmonic.

36. AB are two points on a conic and O is the pole of AB. P is any point on AB, and the tangents therefrom to the conic cut the tangent at S in Q and R. Shew that OQ and OR are conjugate lines.

37. A series of conics pass through two fixed points A, B, and touch a given line l at a fixed point O. Prove that the locus of the poles of AB is a straight line which passes through O and is the harmonic conjugate of l with respect to OA and OB.

38. ABCD is a quadrilateral described about a conic and O is any point. If OP and OQ be the tangents from O, prove that $O[PAQB] = O[PCQD]$.

39. Two conics touch at O and meet again in A and B. U and V are the poles of AB. OPQ is a chord cutting the conics in P and Q respectively. Shew that UP and VQ meet on AB.

40. Two conics meet in the four points A, B, C, D. P is

a point on the first, and PA, PB meet the second in Q and R respectively. Shew that QR and the tangent at P meet on CD.

41. Two conics touch at O and meet again in A and B. OP_1Q_1 and OP_2Q_2 are two chords meeting the first conic in P_1 and P_2, and the second in Q_1 and Q_2. Shew that P_1Q_1 and P_2Q_2 meet on AB.

42. O is a fixed point. Q is a point (1) on a line or (2) on a conic, and R is a collinear point on a fixed line. P is a point on OQR such that [OPQR] is constant. Shew that the locus of P is (1) a line or (2) a conic.

43. A series of conics are drawn touching the two fixed lines OX and OY at the fixed points A and B. V is a fixed point. Prove that the locus of the points of contact of tangents from V to these conics is a conic passing through A and B and having VO and AB for conjugate lines.

44. If two conics cutting in PQRS be such that the tangents at P to each are harmonically conjugate to PR and PS, shew that the tangents at Q are harmonically conjugate to QR and QS.

45. A family of conics have double contact with a given conic at the extremities of a given chord. Prove that the locus of the centres of conics of the family is the diameter of the given conic conjugate to the given chord. (Trinity.)

46. If two conics S_1 and S_2 cut one another in PQRS and be such that the tangents at P harmonically separate PR and PS, then the pole of RS with respect to S_1 is the pole of PQ with respect to S_2.

47. If the polar of O meets a conic in P and Q, and any other line through O meets the conic in A and B, then the four points [PQAB] subtend at any point on the conic a harmonic pencil.

48. O is a point on a conic, OP is the point where the normal at O again meets the conic. OQ and OR are perpendicular chords. QP meets OR in U, and RP meets OQ in V. Prove that UV is a fixed line.

49. ABC is a triangle circumscribing an ellipse, AB touching the conic at D. S is a focus. Shew that the angles ASD, BSC are supplementary. (St. John's.)

50. Two conics cut in P, Q, R, S. A and B are the poles of RS to the conics respectively. AB meets PQ in U. Shew that U[PRAS] is harmonic.

51. Two conics touch at O and meet again in A and B. U is the pole of AB to the first and V is the pole of AB to the second. Prove that O, U, V are collinear, and that OA and OB harmonically separate OUV and the tangent at O.

52. A conic circumscribes a triangle. If P be the point where it cuts a fixed straight line, and if PT be the tangent at P such that P[ABCT] is constant, the conic will pass through another fixed point. (Clare, etc.)

53. P, Q, R are three points in the plane of a conic whose centre is C; through R straight lines RT, RT′ are drawn parallel to the polars of P and Q, intersecting CQ, CP in T and T′ respectively; prove that TT′ is parallel to the polar of R. (Clare, etc.)

54. If three conics have two points in common and each touches each of the other two, the tangents at the points of contact are concurrent. (Clare, etc.)

55. Three conics meet in P, Q, R, S. Chords are drawn through P meeting the conics in the points U, V, W, and the chord QR in X. Shew that [UVWX] is constant. (Clare, etc.)

56. A is a fixed point and B and C are two fixed points on a conic. AP is a variable line meeting BC in M; and L is a variable point on BC such that AL and AP are harmonically conjugate to AB and AC. AL meets the tangent at P in F. LP and FM meet in Q. Prove that the locus of Q is a straight line.

57. Prove that we can project any two given angles into angles of given magnitudes, and at the same time any given line to infinity.

58. Shew how to project any four given points into the corners of a parallelogram of given size and shape.

59. Two parabolas have parallel axes. Prove that their common chord bisects their common tangents.

60. OP_1Q_1 and OP_2Q_2 are two variable chords to a conic through a fixed point O. Shew that the intersection of P_1P_2 and Q_1Q_2 lies on a fixed straight line.

61. Through a fixed point O is drawn a chord cutting the ellipse in P and Q, and on it a point R is taken so that the anharmonic ratio [OPQR] is constant. Prove that the locus of R is a conic having double contact with the ellipse at the points where it is cut by the polar of O. (Clare, etc.)

62. Prove that if two conics have double contact, the polar of the centre of either with respect to the other is parallel to the chord of contact. (Clare, etc.)

63. A tangent to an ellipse at a variable point P meets any two fixed diameters in Q and R. The other tangents to the ellipse through Q and R intersect in T. Shew that the locus of T is a similar and similarly situated ellipse. (Clare, etc.)

64. If PQR is a self-conjugate triangle of a conic, of which ZQX and XRY are chords, shew that YPZ is also a chord. (Math. Trip.)

65. ABC is a triangle inscribed in a conic; PLUQ, PMVR are two chords parallel to BC, CA respectively, L, M lying on AB, U on AC, V on BC. Shew that

LU : UQ :: RV : VM. (Math. Trip.)

66. Three conics S_1, S_2, S_3 have two points A and B in common, and are such that the other chord of intersection of S_2 and S_3 passes through the pole of AB to S_1, and the chord of intersection of S_3 and S_1 through the pole of AB to S_2. Shew that the chord of intersection of S_1 and S_2 passes through the pole of AB to S_3.

67. If PAQ and PBR be chords of a given conic PQR, which pass through given points A and B, and if the chord RS be drawn passing through a given point C collinear with A and B, shew that QS always passes through a certain fixed point.

68. If a circle be described touching an ellipse in P and cutting it in Q and R, and if another circle be described passing through Q and R and touching the ellipse in P′, shew that the tangents at P and P′ are parallel.

69. Two conics S and S′ meet in ABCD. P is any point on S and the lines PA and PB meet S′ in L and M respectively. O is the pole of CD with respect to S. OP and LM meet in T. Prove that OP and LM are harmonically conjugate to TC and TD.

70. AB and CD are two parallel chords in a conic AFBDEC, L and M are their middle points respectively; BM and DL are produced to meet the conic again in E and F; prove that the five following points are collinear, namely, the points of intersection of BM and DL, of BC and DA, of FC and EA, of the tangents at A and C, of the tangents at B and D.

71. The intercept on any tangent to a hyperbola, made by its asymptotes, subtends a constant angle at either focus.

72. From a point P a line is drawn parallel to one asymptote of a hyberbola to meet the curve in T: prove that the line through T parallel to the other asymptote will bisect the chord of contact of tangents from P.

73. A quadrilateral is inscribed in an ellipse; if three of its sides meet a given diameter of the ellipse in fixed points, the fourth side must also meet it in a fixed point.

74. A straight line is drawn through a point O parallel to the asymptote of a hyperbola, to meet the polar of O in Q and the curve in R. Prove that OQ is bisected in R.

75. Prove that the envelope of chords of a conic which subtend a constant angle at a fixed point on the curve is a conic which subtends twice that angle at the point.
(Trinity.)

76. If two fixed conics cut in two points AB, and a series of variable conics be drawn touching the first conic in P and the second conic in Q, and also passing through A and B, prove that PQ passes through one or other of two fixed points U, V. Also if X and Y be the poles of the line AB with respect to the two fixed conics, X, Y, U, V are collinear and [XUYV] is harmonic.

77. Through a fixed point chords of a parabola are drawn. Shew that the locus of their middle points is a parabola. (Pembroke, etc.)

78. Given two points and the focus of a conic, prove that the locus of the other focus is a conic.

79. If l be the polar with regard to a conic of the point through which all chords pass which subtend a right angle at a point P on the conic, prove that the points in which two lines through any point on the conic meet l and those in which the lines meet the conic subtend equal angles at P. (Clare.)

80. ABCDE is a pentagon described about a conic, touching it at P, Q, R, S, T so that P is on AB, Q on BC, and so on; prove that the points of intersection of BC and ED, BS and EQ, BT and EP all lie on a line through A.
(St. Catharine's.)

81. P_1P_2 is a chord of a conic which subtends a right angle at another point P on the conic. The tangents at P and P_1 to the conic meet the tangent at P_2 in H and K respectively; prove that the angle HPK is bisected by the line PP_2. (Queens'.)

82. Two parabolas have their axes parallel and are met by a line parallel to the axes in the points P and Q; prove that the tangents at P and Q meet on the common chord.
(St. Catharine's.)

83. If PR and SQ are parallel chords of a conic such that PQ and RS meet at a point X within the curve, prove that the tangents at P and S meet on a line drawn through X parallel to PR. (St. Catharine's.)

84. OQ and OQ′ are two tangents to a parabola. Another tangent meets OQ and OQ′ in R and R′ respectively. Prove that the tangent parallel to the chord QQ′ bisects RR′.
(Gonville and Caius, etc.)

85. BB′ is the minor axis of an ellipse and BP, BQ any two perpendicular chords through B; shew that BP, B′Q intersect on a fixed straight line, and that if the tangent at P meet this line in Z and meet the tangent at B in X, BP bisects the angle ZBX. (King's, etc.)

86. Tangents to a hyperbola at points P and P′ meet the diameters CP′ and CP in R and R′. Prove that RR′ is parallel to PP′. (St. John's.)

87. Straight lines are drawn parallel to the asymptotes through V, the middle point of a chord PP′, meeting the hyperbola in Q and Q′; prove that QQ′ is parallel to PP′. (Sidney Sussex.)

88. Find the locus of the centre of a hyperbola which passes through two fixed points and whose asymptotes are parallel to two given lines. (Clare, etc.)

89. In any conic chords AB, CD meet in O; tangents at A, C meet in T, tangents at B, D meet in T′; shew that TOT′ is a straight line. (Clare, etc.)

90. If AOA′, BOB′, COC′ are chords of a conic, and P any point on the conic, then the points of intersection of the straight lines BC, PA′, of CA, PB′, and of AB, PC′ lie on a straight line through O. (Trinity, etc.)

91. Determine a point at which three collinear segments subtend equal angles. (Trinity.)

92. If two fixed points on a parabola be given and the direction of the axis, the focus lies on a fixed hyperbola. (Queens'.)

93. Prove that the locus of the vertex of a triangle which circumscribes a conic, and whose base subtends a constant angle at the focus, is a conic. (Downing.)

94. If the extremities of two diagonals of a complete quadrangle are conjugate points with respect to a given conic, the extremities of the third diagonal will also be conjugate points with respect to the same conic. (Trinity.)

95. All chords of an ellipse whose middle points are on the same straight line touch a parabola. (Clare, etc.)

96. If S and S′ be two conics having PQ as a common chord, and if R be any point on PQ, then the polars of R to S and S′ meet on PQ.

97. The asymptotes of one hyperbola are parallel to those of another. Prove that if the curves meet in real

points, the common chord is parallel to one of the diagonals of the parallelogram formed by the asymptotes.
(St. Catharine's.)

98. TP, TQ are any two tangents and TRS any chord of an ellipse. If V is the middle point of RS and QV meet the ellipse again in P′, prove that PP′ is parallel to ST.
(Downing.)

99. If ABCD be four points on a conic such that A, B are harmonically conjugate to C, D, then the intersection of AC, BD, that of AB, CD, that of the tangents at A and D and that of the tangents at B and C are collinear. (Trinity, etc.)

(*N.B.* This means that the chords AB and CD are conjugate lines.)

100. From points on a given straight line, lines are drawn parallel to the polars of the points with regard to a conic; prove that these lines envelope a parabola.
(Trinity, etc.)

101. Prove that the envelope of chords of an ellipse that subtend a right angle at the centre is a concentric circle.
(Trinity.)

102. The envelope of chords of a conic whose middle points lie on a given straight line is a parabola whose axis is parallel to the diameter of the conic conjugate to the given straight line. (Math. Trip.)

103. TP, TP′ are tangents to a parabola, and the diameter through T cuts the curve at Q. If PQ, P′Q cut TP′, TP respectively in R, R′ and the diameters through R, R′ cut the curve in V, V′ respectively, prove that PV, P′V′ intersect on TQ. (Math. Trip.)

104. A point and its polar with respect to a variable circle being given, the polar of any other point A passes through a fixed point B. (Math. Trip.)

105. Two conics are inscribed inside the triangle ABC and both touch AB at the same point. Shew that the common chord of the conics and the point of contact divide that side harmonically. (Clare, etc.)

CHAPTER III.

RECIPROCATION. THE PRINCIPLE OF DUALITY.

1. *Reciprocation.*

Reciprocation is a geometrical process whereby from a *known* proposition about *points* and *lines* is derived mechanically a new proposition about *lines* and *points*.

2. *Notation.*

Let **U** be a given fixed conic.

Let the Polar Line of the point **A** with respect to **U** be denoted by a'.

Let the Pole of the line a with respect to **U** be denoted by **A**$'$.

3. *Process of Reciprocation.*

Let a definite, fixed, conic **U** be given,—to be called the "base-conic of the reciprocation."

If a figure be assigned, we form what is called the Reciprocal Figure by replacing each Point **A** in the original figure by its Polar a' with respect to **U**, and each Line a in the original figure by its Pole **A**$'$ with respect to **U**.

4. *Distinction between a Curve regarded as a Locus and a Curve regarded as an Envelope.*

If a Curve be regarded as a Locus, it is regarded as made up of Points, thus:

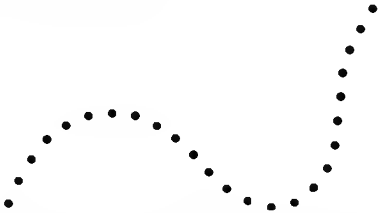

FIG. 34.

If a Curve be regarded as an Envelope, it is regarded as made up of Lines, thus:

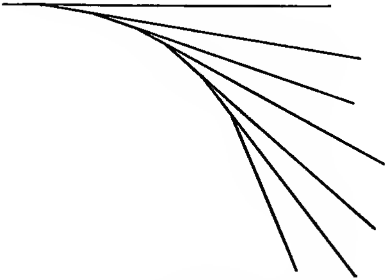

FIG. 35.

In the first figure the line joining two consecutive generating points is a tangent to the curve.

In the second figure the intersection of two consecutive generating lines is a point on the curve, and each generating line is a tangent.

COROLLARY. *The reciprocal of a Locus is an Envelope. The reciprocal of an Envelope is a Locus.*

5. *Just as a Line may be regarded as a Locus of Points, so a Point may be regarded as an Envelope of Lines.*

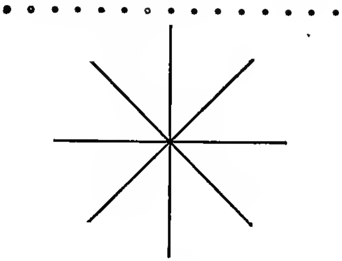

FIG. 36.

The figure explains the idea just enunciated.

6. *The Reciprocal of a Straight Line regarded as a Locus of Points is a Point regarded as an Envelope of Straight Lines.*

Let **A** be a point on the given line l with respect to **U**.

Let **L′** be the Pole of the given line l.

Then, since the Polar of **L′** (*i.e.* l) passes through **A**,

∴ the Polar of **A** passes through **L′**, by Chap. II. Art. 1.

Hence the Polar of every point on l passes through **L′**.

We thus get the following ideas about a Straight Line:

The Reciprocal of a Straight Line regarded as a Generating Element is a point (viz. its Pole with respect to **U***).*

The Reciprocal of a Straight Line regarded as a Locus of Points is a Point (viz. its Pole with respect to **U***) regarded as an Envelope of Straight Lines.*

7. *The Reciprocal of a Point regarded as a Generating Element is a Straight Line (viz. its Polar with respect to* **U***).*

The Reciprocal of a Point regarded as an Envelope of Straight Lines is a Straight Line (*viz. its Polar with respect to* **U**) *regarded as a Locus of Points.*

This will be evident from the method of last article, if we take a point **L** and consider the Poles **A**′, **B**′, **C**′, etc., of all lines a, b, c, etc., passing through **L**.

8. *The Reciprocal of three Collinear Points is three Concurrent Lines.*

The Reciprocal of three Concurrent Lines is three Collinear Points.

This is evident from what has just been discussed.

Corollary. *The join of two points reciprocates into the intersection of two lines and vice versa.*

9. *The Reciprocal of four Collinear Points is four Concurrent Lines having the same Cross-Ratio as the four Collinear Points.*

Let **U** be the base-conic of the reciprocation.

Let the four points **A**, **B**, **C**, **D** lie on the line l, and let the four Polars a', b', c', d' with respect to **U** pass through **L**′, the pole of l.

Project the two points in which the line l meets **U** into the Circular Points at Infinity, whence the proposition is evident.

Corollary. *The reciprocal of four concurrent lines is four collinear points having the same cross-ratio as the four concurrent lines.*

10. *The Principle of Duality.*

Whenever a figure in geometry is given, we can, by choosing a fixed "base-conic of reciprocation" **U**, derive *by a purely mechanical process* a new figure whereby each point of the first figure is replaced by a line in the second figure, and each line in the first figure is replaced by a point in the second figure.

Every known geometrical property of the first figure has a corresponding geometrical property in the second figure.

Hence in proving a geometrical proposition we really prove two propositions.

Such pairs of propositions are called "Dual Propositions," and the principle of proving such two corresponding propositions simultaneously is called the "Principle of Duality."

We give a few examples of this process.

N.B. In general, the process of reciprocation is best performed without using *explicitly and consciously* the "base-conic of the reciprocation," which only serves to complicate matters after the fundamental theorems of the transformation are established.

11. Example 1.

The Reciprocal of the Complete Quadrangle is the Complete Quadrilateral.

ORIGINAL FIGURE.

P, **Q**, **R**, **S** *are four given points.*
The lines **PQ** *and* **RS** *meet in the point* **A**.
The lines **PR** *and* **SQ** *meet in the point* **B**.
The lines **PS** *and* **QR** *meet in the point* **C**.

Re-write the above description, replacing at each step a point by a line; a line by a point; an intersection of straight lines by a join of two points; a join of two points by an intersection of two straight lines, etc.

DUAL (OR RECIPROCAL) FIGURE.

p, q, r, s are four given lines.
The points pq and rs lie on the line a.
The points pr and sq lie on the line b.
The points ps and qr lie on the line c.

Example 2.

ORIGINAL PROPOSITION.

X, Y, U *are three collinear points.*
XP_1, XP_2, XP_3 *are three lines meeting in* X.
YP_1, YP_2, YP_3 *are three lines meeting in* Y.
If $X[P_1P_2P_3U] = Y[P_1P_2P_3U]$,
then P_1, P_2, P_3 *are collinear.*

DUAL (OR RECIPROCAL) PROPOSITION.

x, y, u are three concurrent lines.
xp_1, xp_2, xp_3 are three points lying on x.
yp_1, yp_2, yp_3 are three points lying on y.
If $x[p_1p_2p_3u] = y[p_1p_2p_3u]$,
then p_1, p_2, p_3 are concurrent.

Example 3.

ORIGINAL PROPOSITION.

ABC *is a triangle.*
L *is a fixed point on* AB.
O *is a variable point on* CL.
AO *meets* CB *in* P.
BO *meets* CA *in* Q.
PQ *intersects* AB *in a fixed point* M.
[ALBM] *is harmonic.*

DUAL (OR RECIPROCAL) PROPOSITION.

abc is a triangle.

l is a fixed line through ab.

o is a variable line through cl.

ao is joined to cb by p.

bo is joined to ca by q.

pq is joined to ab by a fixed line m.

[albm] is harmonic.

Example 4.

ORIGINAL PROPOSITION.

ABC *and* DEF *are two triangles.*

If AD, BE, CF *are concurrent,*

then are the intersections of BC, EF *and* CA, FD *and* AB, DE *collinear.*

DUAL (OR RECIPROCAL) PROPOSITION.

abc and def are two triangles.

If ad, be, cf are collinear,

then are the lines joining bc, ef and ca, fd and ab, de concurrent.

12. *Reciprocation of a Given Curve.*

A given curve is regarded as a Locus when it is regarded as generated by points obeying a given law.

A given curve is regarded as an Envelope when it is regarded as generated by straight lines obeying a given law.

Reciprocal of a given curve S *regarded as a Locus.*

We take the polar lines (with respect to the base-conic of the reciprocation U) of the series of points constituting S, regarded as a Locus, and consider the Envelope traced out by these polar lines.

Reciprocal of a given curve S *regarded as an Envelope.*

We take the poles (with respect to the base-conic of the reciprocation U) of the series of lines constituting S, regarded as an Envelope, and consider the Locus traced out by these poles.

To prove that the reciprocal of a given curve S *is the same curve whether* S *be regarded as a Locus or as an Envelope.*

Consider two points A and B lying very near one another on S regarded as a Locus.

Let the polar lines of A and B with respect to the base-conic of the reciprocation be a' and b'.

Then, a', b', c', etc., each form ultimately a portion of the reciprocal curve regarded as an envelope.

But plainly the point $a'b'$ lies ultimately on this envelope.

Now $a'b'$ is the pole of the straight line AB.

Hence the poles of the straight lines AB, BC, CD, etc., lie on the derived curve.

Hence the reciprocal curve S′ *may be considered as made up of the poles of the lines generating* S *regarded as an Envelope, or the polars of the points generating* S *regarded as a Locus.*

13. *If* S *and* S′ *be reciprocal curves, then if any arbitrary line x meets* S *in n points, n tangents can be drawn from any arbitrary point* X′ *to* S′ *and vice versa.*

Regard x and S as each generated by points having n generating points in common.

Then X′ and S′ must each be regarded as generated by lines having n generating lines in common.

COROLLARY I. *The reciprocal of a conic* S *regarded as a locus is a conic* S′ *regarded as an envelope.*

For, since S regarded as a locus meets an arbitrary line in two points, hence two tangents can be drawn to S′ regarded as an envelope from an arbitrary point, which is an exclusive property of conics. Thus S′ is a conic.

COROLLARY II. *The reciprocal of a conic* S *regarded as an envelope is a conic* S′ *regarded as a locus.*

COROLLARY III. *The reciprocal of a point* L *on the conic* S *is a tangent* l' *to the conic* S′.

COROLLARY IV. *The reciprocal of a tangent* l *to the conic* S *is a point* L′ *on the conic* S′.

COROLLARY V. *The Base-Conic of the reciprocation* U *reciprocates into itself; a point on* U *into the tangent at that point; and a tangent to* U *into the point of contact of that tangent.*

14. *The Reciprocal of a Pole and Polar with respect to the Conic* S *is a Polar and Pole with respect to the Conic* S′.

The tangents at the points B and C on the conic S are c and b respectively.

b and c intersect in A.

The line joining B and C is a.

Then A is the pole of a.

DUAL (OR RECIPROCAL) FIGURE.

The points of contact of the tangents b' and c' to the conic S' are C' and B' respectively.

B' and C' are joined by a'.

The intersection of b' and c' is A'.

Then a' is the polar of A'.

SELF-CONJUGATE TRIANGLE. A triangle is said to be self-conjugate with respect to a conic S when each side is the polar of the opposite vertex.(32)

COROLLARY I. *The reciprocal of "the vertices of a triangle self-conjugate to the conic* S*" is "the sides of a triangle self-conjugate to the conic* S'*."*

COROLLARY II. *The reciprocal of "the sides of a triangle self-conjugate to the conic* S*" is "the vertices of a triangle self-conjugate to the conic* S'*."*

15. *A point of intersection of two conics* S_1 *and* S_2 *reciprocates into a common tangent of* S'_1 *and* S'_2.

Let the four points of intersection of the two conics S_1 and S_2 be P, Q, R, S.

Let the four common tangents of the reciprocal conics S'_1 and S'_2 be p', q', r', s'.

Since P lies on S_1, $\therefore p'$ touches S'_1.

Since P lies on S_2, $\therefore p'$ touches S'_2.

Hence p' touches both S'_1 and S'_2.

COROLLARY I. *The fact that two conics* S_1 *and* S_2 *intersect in four points reciprocates into the fact that two conics* S'_1 *and* S'_2 *have four common tangents.*

FOUR-POINT SYSTEM OF CONICS. The singly-infinite system of conics that can be drawn through four given points is called a Four-Point System of Conics.(33)

Four-Line System of Conics. The singly-infinite system of conics that can be drawn to touch four given lines is called a Four-Line System of Conics.(34)

Corollary II. *A Four-Point System of Conics reciprocates into a Four-Line System of Conics and vice versa.*

16. *Two conics* S_1 *and* S_2 *touching one another reciprocate into two conics* S'_1 *and* S'_2 *touching one another.*

Let S_1 and S_2 have in common the two coincident points P and Q.

$\therefore$ S'_1 and S'_2 have in common the two coincident tangents p' and q', whence it is evident that S'_1 and S'_2 touch at the ultimate point of intersection of p' and q'.

Corollary. *Two conics having double contact reciprocate into two conics having double contact.*

17. *Degenerate Conics.*

We are accustomed in dealing with Conic Loci to regard a pair of straight lines as a degenerate conic. Now two straight lines reciprocate into two points. Also, since two straight lines belong to the treatment of conics as loci, and since the reciprocals of Loci are Envelopes (Chap. III. Art. 4, Cor.), hence *the degenerate case of a conic regarded as an envelope is a couple of points.*

18. Example 1.

To find how many conics can be drawn to touch two given lines and to pass through three given points.

The Dual Theorem is easier to solve, viz.:

To find how many conics can be drawn to pass through two given points and touch three given lines.

The answer is four (Chap. II. Art. 23, Ex. 2).

Example 2.

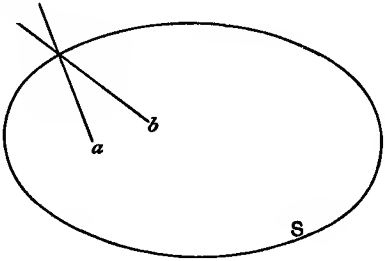

FIG. 37.

Reciprocate the following figure:

The description of the figure is, *a and b are two lines whose point of intersection lies on the conic* S.

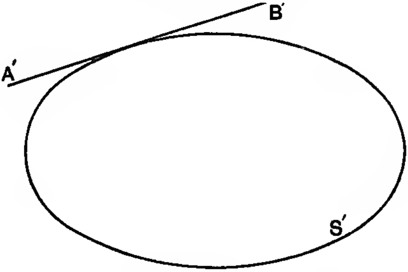

FIG. 38.

The description of the Dual (or Reciprocal) Figure would therefore be, A′ *and* B′ *are two points whose join touches the conic* S′.

Example 3.

Reciprocate the figure:

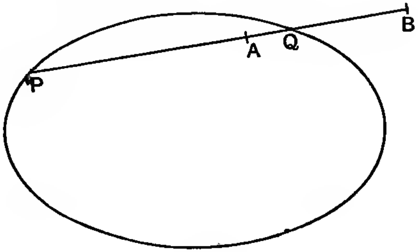

FIG. 39.

Description of Original Figure.

The join of the two points A, B *cuts the conic* S *in two points* P, Q *that harmonically separate* A, B.

Dual (or Reciprocal) Figure.

The intersection of the two lines a', b' *lies on two tangents* p', q' *that harmonically separate* a', b'.

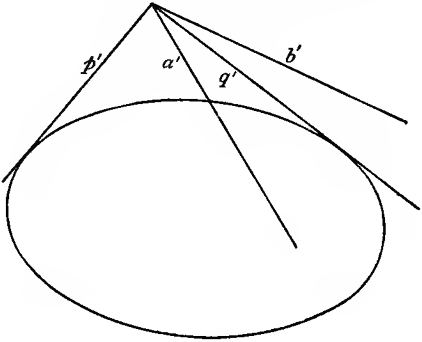

FIG. 40.

Example 4.

Prove that the polars of the point pq *with respect* *to all conics touching the four fixed lines* p, q, r, s *pass through a fixed point.*

The Dual Theorem is easier to prove, viz.:

Prove that the poles of the line $P'Q'$ *with respect to all conics passing through the four fixed points* P', Q', R', S' *lie on a fixed line.*

Project P', Q' into the Circular Points at Infinity, and the result is evident.

Example 5.

ABC *is a fixed triangle. A series of conics inscribed to this triangle pass through a fixed point* L. *Prove that the polar of* C *envelopes a conic of which* L *and* AB *are pole and polar.*

The Dual Theorem is more easily proved, viz.:

$a'b'c'$ *is a fixed triangle. A series of conics circumscribed to this triangle touch a fixed line* l'. *Prove that the pole of* c' *describes a conic of which* l' *and* $a'b'$ *are polar and pole.*

Project $b'c'$ and $c'a'$ into the Circular Points at Infinity, and the result required reduces easily to finding the locus of a point equally distant from a fixed point and a fixed line.

N.B. In the above examples, the dashes might have been omitted from the letters in the description of the reciprocal figure, as there is no chance of confusion between the derived and the original figures.

19. *Case when a Circle is the Base-Conic of the Reciprocation.*

If P be a point and p' its polar with respect to the circle centre O, we know that (if OP meets p' in N),

(1) p' is perpendicular to OP;

(2) ON . OP = (radius)^2.

Thus a simple geometrical construction is found for the pole and polar, when a circle is the base-conic of the reciprocation.

O is called the Centre of Reciprocation.

Corollary I. *The reciprocal of the centre of reciprocation in the original figure is the Line at Infinity in the derived figure.*

Corollary II. *The reciprocal of the Line at Infinity in the original figure is the centre of reciprocation in the derived figure.*

Corollary III. *The reciprocal of a point on the base-circle of the reciprocation is the tangent at that point.*

Corollary IV. *The reciprocal of a tangent to the base-circle of the reciprocation is the point of contact.*

Corollary V. *The base-circle reciprocates into itself.*

Corollary VI. *The reciprocal of a circle concentric with the base-circle is another circle concentric with the base-circle.*

Corollary VII. *The reciprocal of a line p passing through* O, *the centre of reciprocation, is the point* P′ *on the line at infinity, such that* OP′ *is perpendicular to p.*

Corollary VIII. *The angle subtended by any two points* P *and* Q *at the centre of reciprocation is equal to the angle between their reciprocal lines* p' *and* q'.

Corollary IX. *The angle between any two lines* p *and* q *is equal to the angles subtended by their reciprocal points* P′ *and* Q′ *at the centre of reciprocation.*

Corollaries VIII. and IX. are important in what follows.

20. The following fundamental transformations in reciprocating with respect to a circle are to be noted:

Let S denote the original curve and S′ the reciprocal curve.

Let O denote the centre of reciprocation.

I. *If* S *pass through* O, S′ *touches the Line at Infinity.*

II. *If* S *pass through* O, *the tangent to* S *at* O *reciprocates into the point of contact of* S′ *with the Line at Infinity.*

III. *A tangent from* O *to* S *reciprocates into a point of intersection of* S′ *with the Line at Infinity.*

IV. *A point of intersection of* S *with the Line at Infinity reciprocates into a tangent from the centre of reciprocation to* S′.

V. *The reciprocal of* I (*i.e. one of the Circular Points at Infinity*) *is* OI.

For I lies on the base-circle, and the reciprocal of I is therefore the tangent at I to the base-circle, *i.e.* OI.

VI. *The reciprocal of a conic with respect to a base-circle of reciprocation is a conic.*

The proof of this is contained in Chap. III. Art. 13, Cors. I. and II.

VII. *The reciprocal of the centre of a conic* S *is the polar of* O *with respect to* S′.

For pole and polar with respect to S reciprocate into polar and pole with respect to S′.

Now the centre of S is the pole of the Line at Infinity with respect to S.

The Line at Infinity reciprocates into O.

Hence the centre of S reciprocates into the polar of O with respect to S′.

VIII. *The asymptotes of* S *reciprocate into the points of contact of the tangents from* O *to* S′.

IX. *A Rectangular Hyperbola* S *reciprocates into a conic* S′ *such that the tangents from* O *to* S′ *are perpendicular.*

This is an immediate corollary from VIII.

X. *If* O *be situated on the director-circle of* S, S′ *is a Rectangular Hyperbola.*

This is a convenient way of stating IX.

XI. *If* S *be a conic passing through* O, S′ *is a parabola.*

21. We see, by Chap. III. Art. 13, that the reciprocal of a conic with respect to a base-circle is a conic. If, however, we choose a focus of the conic to be reciprocated as the centre of reciprocation, the conic is transformed into a circle. Thereby many important properties of conics are easily deduced.

Any conic can be reciprocated into a circle by choosing a focus as the centre of reciprocation.

We shall give two proofs of this depending on entirely different principles.

First Proof.

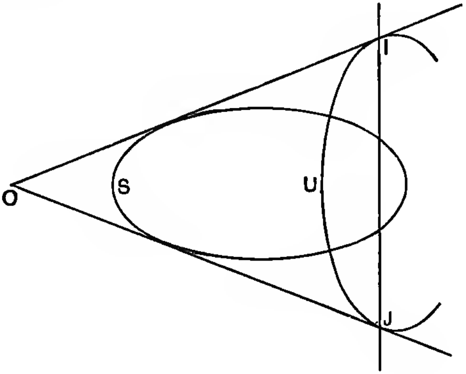

FIG. 41.

Let **O** be the focus of the conic **S**, and let **U** be the "base-circle of the reciprocation" having **O** as centre.

Let us interpret all the lines and points of the above figure as explained in Chap. II.

By Chap. III. Art. 13 the reciprocal of the conic **S** with respect to **U** is another conic **S′**.

Since **OI** touches **U** at **I**, the reciprocal of **OI** with respect to **U** is the point **I**.

But the reciprocal of a tangent to **S** with respect to **U** is a point on **S′**, hence **I** lies on **S′**.

Similarly **J** lies on **S**.

Thus **S′** is a circle.

Second Proof.

Let O be a focus of the given conic.

Let OY, the perpendicular from O to the tangent at P, be produced to Q so that $OY . OQ = k^2$ (where k is the radius of the base-circle).

Then Q describes the reciprocal of the given conic with respect to the circle whose centre is O and radius k.

But Y lies on the Auxiliary Circle.

Hence Q describes the Inverse of the Auxiliary Circle with respect to the Centre of Inversion O and Radius of Inversion k.

Hence the locus of Q is a circle.

22. *Conversely, the reciprocal of a circle* S′ *with respect to another circle* U *is a conic having the centre of* U *as focus.*

First Proof.

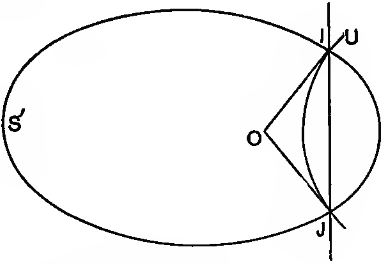

FIG. 42.

Let O be the centre of U.

By Chap. III. Art. 13 the reciprocal of S′ with regard to U is a conic S.

The reciprocal of the point I lying on S′ is the line OI touching S.

Similarly, OJ touches S.

Hence S is a conic having O as focus, by Chap. II. Art. 16.

Second Proof.

We shall require the following proposition in the Geometrical Theory of Conics:

If O be a given point and Y a variable point on a fixed circle, the line through Y perpendicular to OY envelopes a conic with O as focus.(35)

Let Q be a point on the given circle S′ to be reciprocated with regard to a circle U whose centre is O and radius k.

Let Y be a point on OQ such that $OY \cdot OQ = k^2$.

Then Y describes the Circle which is the Inverse of S with regard to the Centre of Inversion O and the Radius of Inversion k.

Hence, if YP be perpendicular to OY, YP will envelope a conic with O as focus by the proposition just quoted.

23. *The Reciprocal of a Circle* S′ *with regard to a Circle* U, *whose Centre is* O, *is an Ellipse, Hyperbola or Parabola according as* O *the Centre of Reciprocation is within, without or on the circumference of the Circle* S′ *to be reciprocated.*

Let O the centre of reciprocation be outside the Circle S′ to be reciprocated.

Let p and q be the tangents from O to S′.

Then the poles of p and q with respect to a circle centre O are plainly at infinity and are the points in which the reciprocal curve S cuts the Line at Infinity, and are in this case real, since p and q are real when O is outside.

The points in which S *cuts the Line at Infinity are real if* O *lie outside* S′, *i.e.* S *is a Hyperbola.*

The points in which S *cuts the Line at Infinity are imaginary if* O *lie inside* S′, *i.e.* S *is an Ellipse.*

If O lie outside S′ and very close to it, it is plain that p and q are nearly coincident. Hence the points in which S cuts the Line at Infinity are nearly coincident.

Hence, *if* O *lie on the circumference of* S′, S *is a parabola.*

COROLLARY I. *If* C′ *be the Centre of* S′, *the circle to be reciprocated,* C′ *reciprocates into the Directrix of* S.

For pole and polar reciprocate into polar and pole (Chap. III. Art. 14).

Now the Line at Infinity in the original figure reciprocates into O.

Let C′ reciprocate into the line c.

Since the Line at Infinity is the Polar of C′ with respect to S′, ∴ O is the Pole of c with respect to S, *i.e.* c is the polar line of the focus O, and is therefore the directrix of S.

COROLLARY II. *The line* V′N′W′, *which is the polar of* O *with respect to* S′, *reciprocates into the Centre of* S.

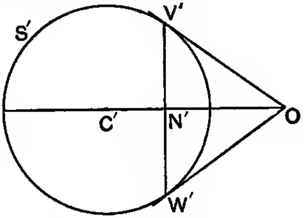

FIG. 43.

For O reciprocates into the Line at Infinity.

Let **V′N′W′** reciprocate into the point **N**.

Since **O** is the pole of **V′N′W′** with respect to **S′**, ∴ the Line at Infinity is the polar of **N** with respect to **S**, *i.e.* **N** is the centre of **S**.

COROLLARY III. *The points* **V′** *and* **W′** *in the figure to Corollary II. reciprocate into the Asymptotes of* **S**.

Since **V′**, **W′** are the points of contact of the tangents from **O**, therefore v, w will be the tangents at the points of intersection of the Line at Infinity with **S**.

COROLLARY IV. *If* **OV′** *and* **OW′** *be perpendicular,* **S** *will be a Rectangular Hyperbola.*

For **S** cuts the Line at Infinity at the points in which the perpendiculars through **O** to **OV′** and **OW′** cut the Line at Infinity; and since **OV′** and **OW′** are perpendicular, the lines through **O** perpendicular to them will also be perpendicular. Thus **S** is a Rectangular Hyperbola.

COROLLARY V. *A system of Confocal Conics reciprocates with respect to one of their foci into a system of Coaxal Circles.*

Let **O** be the focus which is chosen as the centre of reciprocation.

Then, by Chap. III. Art. 21, each conic of the system reciprocates into a circle.

Now the centre of the conic **S** reciprocates into the polar of **O** with respect to **S′**.

Hence, since the confocal conics have a common centre, their reciprocal circles will have the same polar with respect to **O**.

Thus the circles will be a coaxal system having **O** as one limiting point, and the above common polar of O will pass through the other limiting point.

Otherwise. A confocal system of conics all touch the same four tangents from I and J.

They will therefore reciprocate into a system of four-point conics passing through the poles of OI and OJ with respect to the Base-Circle U, *i.e.* through I and J.

They are, therefore, a system of coaxal circles.

N.B. Many properties of Confocal Conics can thus be quite easily derived from those of Coaxal Circles.

24. Example 1.

If A *and* B *be two fixed points, and* P *a moving point such that* PA *is perpendicular to* PB, *the locus of* P *is a circle* S.

Reciprocate with respect to O the middle point of AB.

We thus get the following figure:

a' is the polar of A and is therefore fixed.

b' is the polar of B and is therefore fixed.

p' is the polar of P and is therefore variable.

Since the centre of reciprocation is also the centre of the circle S to be reciprocated, the reciprocal curve S′ is evidently a concentric circle.

We thus have the following Dual Theorem:

a' and b' are two fixed parallel lines and O *is a point half-way between them. p' is a moving line such that the portion of p' intercepted between a' and b' subtends a right angle at* O. *The envelope of p' is a circle whose centre is* O.

Example 2.

Let us reciprocate the above proposition with **B** as the centre of reciprocation.

a' is the polar of **A** and is fixed.

b' is the polar of **B** and is the Line at Infinity.

p' is the polar of **P** and is variable.

Since the centre of reciprocation lies on the circle **S**, the reciprocal curve is a parabola.

Since **PA** is perpendicular to **PB**, the points $p'a'$ and $p'b'$ subtend a right angle at **B**, *i.e.* p' is perpendicular to the join of $p'a'$ and **B**.

Thus we get the Dual Theorem:

If a variable line p' move so that p' is perpendicular to the join of a fixed point **B** *with the point of intersection of a fixed line a' with p', then will p' envelope a parabola.*

This result is important and should be remembered.

Example 3.

The angle **APB** *in the same segment of a circle is constant in magnitude.*

Dual Theorem (any circle being chosen as the base-circle of the reciprocation).

Let **O** be the centre of the base-circle of the reciprocation.

If a' and b' be two fixed tangents to a conic with focus **O**, *and if p' be a variable tangent, then the points $p'a'$ and $p'b'$ subtend a constant angle at* **O.**

Example 4.

Any diameter **AXB** *of a circle* **S** *is bisected at the centre* **X.**

Let O be the centre of the base-circle of the reciprocation. Then the given circle S becomes a conic with focus O. Let AXB meet the Line at Infinity in Y. Then A, B reciprocate into a', b' tangents to the conic. Also X reciprocates into x', directrix to the conic. And Y reciprocates into y', a line through O.

Since A, X, B, Y are collinear and harmonic,

$\therefore$ a', x', b', y' are concurrent and harmonic.

Dual Theorem.

The two tangents drawn from any point on the directrix of a conic to that conic harmonically separate the directrix and the line joining that point to the focus.

Example 5.

The common chord of two circles is perpendicular to the line joining their centres.

Let O be the centre of the base-circle of the reciprocation.

Let the centres of the two circles be A and B, and their points of intersection H and K.

Then the two circles reciprocate into two conics, each with focus O.

The points of intersection H, K reciprocate into common tangents h', k'.

The centres A, B reciprocate into the directrices a', b'.

Since HK is perpendicular to AB, $\therefore$ $h'k'$ and $a'b'$ subtend a right angle at O.

Dual Theorem.

The point of intersection of the directrices and the point of intersection of the two common tangents of *two conics with a common focus subtend a right angle at that focus.*

Example 6.

The rectangle contained by the segments of all chords of a circle passing through a given point O *is constant.*

Let POQ be a chord through the fixed point O.

Let O be taken as the centre of reciprocation.

Let k be the radius of the base-circle of reciprocation.

Let the given circle S be reciprocated into the conic S′.

Produce OP to Y such that $OP \,.\, OY = k^2$. ....(1)

Produce OQ to Z such that $OQ \,.\, OZ = k^2$. ....(2)

Then the lines through Y and Z perpendicular to PQ touch S′.

By (1) and (2), $OY \,.\, OZ = \dfrac{k^4}{OP \,.\, OQ} = \text{constant}$.

DUAL THEOREM.

The product of the perpendiculars from a focus on two parallel tangents to a conic is constant.

Example 7.

Two parabolas, with concavities turned opposite ways, have parallel directrices, and a common focus O. *Prove that the tangents at the four real points, in which the two directrices cut the parabolas, touch a conic of which* O *is one focus.* (St. John's.)

The reciprocal theorem (with respect to O) is easier to solve. It may be stated as follows:

Two circles, turned in opposite directions, touch at

o. *Prove that the points of contact of the four real tangents drawn from the centres to the circles lie on a circle.*

It is plainly the circle having the line joining the centres as diameter.

25. Many of the examples appended to this chapter will be more easily solved by remembering the two following propositions (already quoted):

I. **If S be a fixed point, x a fixed line, and Y a variable point on x, the straight line through Y perpendicular to SY envelopes a parabola having S as focus and x as the tangent at the vertex.**(36)

II. **If S be a fixed point, C a fixed circle, and Y a variable point on C, the straight line through Y perpendicular to SY envelopes a conic having S as focus and C as auxiliary circle.**(37)

26. We thus see that the process of reciprocation is useful in the two following ways:

(1) We may derive new propositions from given propositions by reciprocating them.

(2) The reciprocal of a given proposition is often easier to solve than the proposition itself.

EXAMPLES III.

Write out a description of each of the following figures, and hence draw the reciprocal figures.

1.

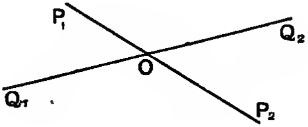

FIG. 44.

2.

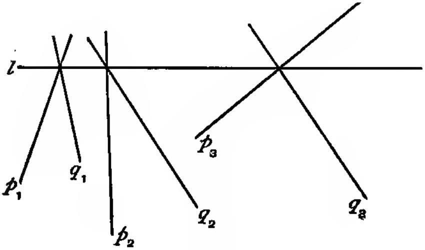

FIG. 45.

3.

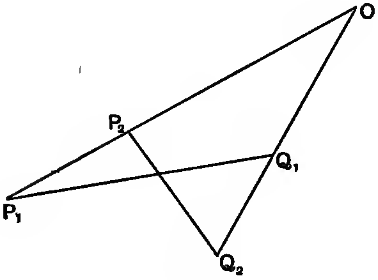

FIG. 46.

4.

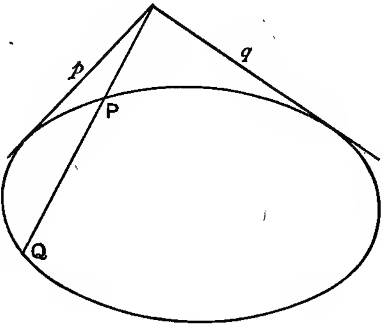

FIG. 47.

5.

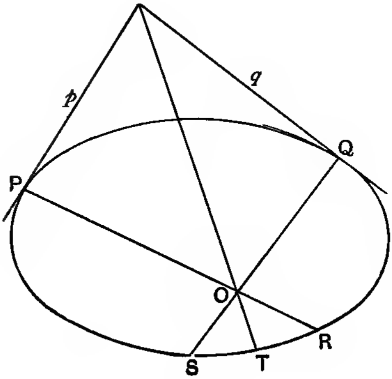

FIG. 48.

6.

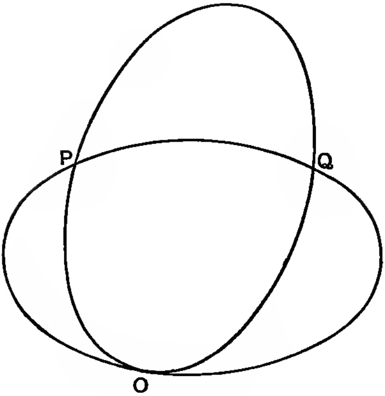

FIG. 49.

7. ABC is a triangle. A series of conics are drawn touching AB at the fixed point K and also touching CA and CB at the variable points P and Q respectively. Prove that PQ passes through a fixed point L in AB.

8. Shew also that in question 7, [AKBL] is a harmonic range.

9. A series of conics touch a fixed line AB at the fixed point O, and also touch the two fixed lines l and m. OX is a fixed line through O meeting a conic of the system again in P. Prove that the tangent at P envelopes a conic.

10. Prove that if S be the conic enveloped in question 9, then S touches AB also.

11. If K be the point where the conic S in question 9 touches AB, shew that O and K harmonically separate L and M, the points where l and m respectively meet AB.

12. Two fixed conics S_1 and S_2 touch each other at O, and both touch the fixed lines l and m. S is a variable conic also touching l and m, and touching S_1 at P_1 and S_2 at P_2. Prove that the intersection of the tangents at P_1 and P_2 lies on a fixed line n.

13. Prove that in question 12, the polars of the point lm with respect to S_1 and S_2 respectively meet on n.

14. Prove also that in question 12, the line P_1P_2 always passes through O, the point of contact of S_1 and S_2 for all positions of S.

15. A series of conics touch a fixed line AB at the fixed point O, and also touch the fixed lines l and m. OX and OY are two fixed lines passing through O, and a conic of the system cuts OX again at P and OY at Q. Prove that the envelope of PQ is a conic S.

16. Prove that in question 15, the conic S touches AB.

17. Prove also that in question 15, S passes through the four points of intersection of OX and OY with l and m.

18. Two conics are inscribed in the quadrangle P, Q, R, S. One of the points of intersection of the two conics is A. Prove that the tangents at A harmonically separate AP; AR and AQ; AS.

19. S_1 and S_2 are two given conics. A common tangent touches S_1 at P and S_2 at Q. Another common tangent touches S_1 at R and S_2 at S. The other two common tangents to S_1 and S_2 are denoted by a and b. Prove that if a and b cut PQ in points harmonically conjugate to P and Q, then a and b will also cut RS in points harmonically conjugate to R and S.

20. A series of conics touch the four sides of the quadrangle P, Q, R, S and cut two fixed lines through P in U and V respectively. Prove that UV envelopes one or other of two fixed conics passing through the four points in which the two fixed lines through P cut QR and RS.

21. l and m are two common tangents to two given conics. Another variable conic touches l and m, and also touches the two given conics at P and Q respectively. Prove that PQ always passes through the intersection of the other two common tangents (*i.e.* not l and m) of the two given conics.

22. Prove also that in question 21, if the tangents at P and Q meet in R, R lies on one or other of two fixed lines concurrent with the polars of lm and harmonically separating them.

23. OX and OY are two fixed lines. A series of conics touch OX, OY and two other fixed lines a and b. OZ is a fixed line through O meeting the variable conic in P and Q. Prove that the tangents at P and Q both envelope the same conic S.

24. In question 23, prove that OZ is the polar of the point ab, with respect to the conic S.

25. In question 23, prove that OX and OY are conjugate lines with respect to S.

26. The two conics S_1 and S_2 are inscribed to the quadrangle KLMN. Any line meets KL in U and KN in V. UP_1 and UP_2 are the tangents other than UK to S_1 and S_2 respectively. VQ_1 and VQ_2 are the tangents other than VK to S_1 and S_2 respectively. Prove that

$$U[P_1P_2MV] = V[Q_1Q_2MU].$$

27. Two fixed conics S_1 and S_2 have one pair of common tangents meeting in T, the points of contact with S_1 being P_1 and Q_1 respectively, and the corresponding points of contact with S_2 being P_2 and Q_2 respectively. A series of variable conics S is drawn touching P_1P_2 and Q_1Q_2 and also the original conics S_1 and S_2. Prove that the polar of T with respect to S envelopes a conic U passing through P_1, P_2, Q_1, Q_2.

28. If S_1 and S_2 touch in question 27, shew that U touches them at their point of contact.

29. In question 27, shew that U touches each of the other pair of common tangents to S_1 and S_2.

30. S is a given conic, A and B are two fixed points on it and the tangents at A and B meet in C. A series of conics is drawn touching CA and CB at P and Q respectively, touching S and passing through a fixed point O. Prove that PQ envelopes a conic U passing through A, B and C.

31. Prove that in question 30, the tangent at O meets the tangent where the two conics touch in a point whose locus is a straight line touching the conic S.

32. A series of circles are drawn passing through the two fixed points A and B. BX is a fixed line through B cutting a circle of the system in P. Prove that the tangent at P to the circle envelopes a parabola having A for focus.

33. A series of circles pass through the two fixed points A and B. Prove that the tangents to the circles at the other ends of the diameters through A envelope a parabola having A for focus and B for vertex.

34. If each of three conics touch two given straight lines, shew that the points of intersection of their other pairs of common tangents are collinear. (Clare, etc.)

35. Shew that of all conics touching four straight lines, one is a parabola.

36. How many conics can be inscribed in a given triangle and have a given point as focus?

37. Shew that two parabolas with the same focus have only two common tangents.

38. A series of conics have the same focus and directrix. A fixed line intersects them. Shew that the envelope of the tangents to the conics at the points where this line meets them will be a conic having the same focus and touching both the fixed line and the common directrix.

39. Prove that, if three conics have a common focus and each pair intersect in four real points, of their chords of intersection there are four groups of three which are concurrent, each of the three being a chord of intersection of a different pair of the conics. (Math. Trip.)

40. Two parabolas with parallel axes have two common tangents.

41. Prove that two conics (real or imaginary) can be drawn which have a given point as focus, touch two given lines and pass through a given point. (Clare, etc.)

42. The envelope of a chord of a circle which subtends a right angle at a given point within the circle is an ellipse, of which one focus is at the centre of the circle and the other at the given point.

43. Three parabolas A, B, C touch the same line l; and B, C have a common focus S. If P and Q are the respective intersections of the pairs of common tangents other than l of A and B and of A and C, prove that S, P, Q are in a straight line. (St. John's.)

44. Two fixed conics have a common focus S, and a variable conic with the same focus touches them in P_1 and P_2. Shew that the line P_1P_2 passes through a fixed point, and the directrix of the variable conic corresponding to the focus S envelopes another conic. (Jesus, etc.)

45. Two conics have a common focus S, and any line through S meets one conic in P and P′ and the other in Q and Q′. Shew that the tangents at P and Q meet on one fixed straight line and those at P and Q′ meet on a second fixed straight line. (Clare, etc.)

46. Two parabolas have their axes at right angles and have a common focus S. Shew, by reciprocation, that the tangents at their vertices intersect on their common tangent, and that the radii vectores from S to the points of contact of the common tangent are at right angles.

(Peterhouse, etc.)

47. A series of confocal parabolas touch a fixed tangent. Through a fixed point on this tangent are drawn the other tangents to the parabolas. Prove that the locus of the point of contact is a circle.

48. Two parabolas have a common focus. From any point on a common tangent two other tangents are drawn to the parabolas. Shew that another parabola with the same focus can be drawn touching the last two tangents and the line joining their points of contact. (Trinity.)

49. Two parabolas have a common focus, and from a point T on their common tangent two other tangents TP, TQ are drawn to the curves. Shew that the angle PTQ is equal to the angles between the axes of the parabolas.

(Clare.)

50. If two conics have a common focus, the points of contact of each of their two common tangents subtend equal angles at that focus.

51. If two intersecting parabolas have a common focus, prove that the angle between their axes is equal to that which their common tangent subtends at the focus.

52. A series of conics all touch a fixed conic S at A and B. Another fixed conic S′ passes through A and B. Prove that the tangents to the variable conics at the points where they meet S′ envelope a conic.

53. From C, the centre of a fixed circle, any radius CP is drawn, the tangent at P meets a fixed line l at I, a line through C perpendicular to CP meets at I′ a fixed line m parallel to l. Prove that II′ touches a fixed conic of which C is a focus and m the corresponding directrix. (Clare, etc.)

54. Prove that two real and different conics cannot be such that the reciprocal polar of each with respect to the other is the same conic.

55. The lines joining the ends of a fixed chord of a circle to the ends of any parallel chord meet on a fixed straight line. Reciprocate this theorem. (Clare, etc.)

56. Three conics A, B, C have the same focus and a common tangent. A, B touch this tangent at X; C touches it at Y. The other common tangent of A, C is KM; the other common tangent of B, C is KN. Any tangent to C meets KM, KN at M, N. Prove that the other tangent from M to A and the other tangent from N to B meet on XY. (Pembroke.)

57. From a fixed point P pairs of tangents PQ, PQ′ are drawn to each of a system of confocal conics. Prove that the circles PQQ′ form a coaxal system. Reciprocate this theorem. (Clare, etc.)

58. A variable tangent to a parabola meets two fixed tangents in P, Q, and the diameters through P and Q meet the curve in L and M. Prove that the envelope of LM is a conic. (St. John's.)

59. A family of parabolas touch two fixed straight lines and pass through a fixed point. Shew that their directrices envelope a conic. (Clare, etc.)

60. A series of ellipses have the same focus, pass through a given point and touch a given line. Shew that their centres lie on a hyperbola. (Trinity.)

CHAPTER IV.

GENERAL PROPERTIES OF CONICS.

1. It is our purpose now to investigate some of the more important general properties of conics.

Throughout this chapter, a conic regarded as a locus will be denoted by S, and a conic regarded as an envelope will be denoted by Σ.

2. *Projective properties of the line bisecting a given line perpendicularly.*

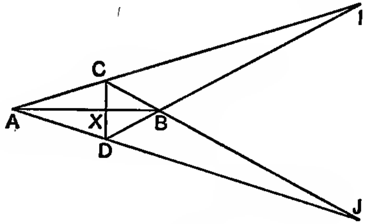

FIG. 50.

Let AB be the given line, A and B being real points.

Let I and J be the Circular Points at Infinity.

Let AI and BJ intersect in C; and AJ and BI intersect in D.

By Chap. I. Art. 14, X is the harmonic conjugate with respect to A, B of the point in which AB cuts the

Line at Infinity IJ; that is, X is the middle point of AB (where A and B are real).

Also X[BICJ] is harmonic (Chap. I. Art. 14), *i.e.* XB and XC are perpendicular (Chap. II. Art. 11).

Hence if A and B are real, CD is the perpendicular bisector of the line AB.

N.B. It can easily be seen from the elementary properties of conjugate imaginaries, that if A and B are real, C and D are conjugate imaginaries, and vice versa.

The result established in this article will often be found useful in dealing with projections where some of the lines in the projected figure pass through the Circular Points at Infinity.

Example.

Two conics have double contact at A *and* B. *The tangent at the variable point* P *to one conic cuts the other curve at* C *and* D. *If* AC *and* BD *meet in* U, *prove that* UP *passes through a fixed point.*

Let the tangents at A and B meet in O.

Project A, B into the Circular Points at Infinity A'_∞, B'_∞. The conics then become two circles having the common centre O'. $C'D'$ is bisected at P'.

Also, by Chap. IV. Art. 2, since $C'A'_\infty$ and $D'B'_\infty$ meet in U', therefore $U'P'$ is the perpendicular bisector of the chord $C'D'$, and hence passes through the centre O'.

Thus UP passes through the fixed point O.

3. The dual theorem of the following proposition is important:

If A, B, C, D *be four fixed points on a given conic and* P *a variable point on that conic, then* P[ABCD] *is constant.*

DUAL THEOREM.

If a, b, c, d *be four fixed tangents on a given conic and* p *a variable tangent to that conic, then* $p[abcd]$ *is constant.*

This last theorem may also be proved directly thus: Project the points of contact of a and b into the Circular Points at Infinity.

We then obtain a circle having the point $a'b'$ as centre and touched by lines c', d'.

We have to prove that the intercept of p' between c' and d' subtends a constant angle at the centre $a'b'$.

Let the points of contact of the tangents c', d', p' be C', D', P' respectively.

Let p' meet c' in C_1' and d' in D_1'.

Let O' be the centre.

Then plainly the angle $C_1'O'P'$ = half the angle $C'O'P'$,
and the angle $D_1'O'P'$ = half the angle $D'O'P'$.

Hence the angle $C_1'O'D_1'$ = half the angle $C'O'D'$
= a constant.

4. FOUR-POINT SYSTEM OF CONICS. A system of conics, all the members of which pass through four given points, is called a Four-Point System of Conics.(38)

Fundamental property of a Four-Point System of Conics.

Let two given conics intersect in the points P, Q, R, S. Let ABC be the Harmonic Triangle of the quadrangle PQRS.

Then, it can be proved that ABC *is a triangle self-conjugate with respect to all conics passing through* P, Q, R, S.

For the line BC passes through the harmonic conjugate of A with respect to P, Q, and also through the harmonic conjugate of A with respect to R, S (Chap. I. Art. 13).

Now the polar of A passes through all the harmonic conjugates of A with respect to those pairs of points in which lines through A cut the conic (Chap. II. Art. 1).

Thus CB is the polar of A, and so on.

5. *Projective Aspect of Coaxal Circles.*

Project R, S into the Circular Points at Infinity R'_∞, S'_∞, in the figure of Art. 4.

We have then to deal with a system of coaxal circles passing through P', Q'.

A degenerate conic of the four-point system P', Q', R'_∞, S'_∞ is the conic consisting of the two lines $C'Q'R'_\infty$ and $C'P'S'_\infty$. Their point of intersection C' is evidently a Limiting-Point (or, viewed metrically, a Point-Circle) of the coaxal system of circles passing through P', Q'.

Similarly, B' is the other Limiting-Point of the Coaxal System.

Also, the Radical Axis $P'Q'$ meets the Line at Infinity in A'.

6. FOUR-LINE SYSTEM OF CONICS.

A system of conics, all the members of which touch four given lines, is called a Four-Line System of Conics.(30)

Fundamental property of a Four-Line System of Conics.

The "diagonal triangle" of the quadrilateral, all of whose sides are touched by the members of a Four-Line System of Conics, is self-conjugate with respect to each of the members of the system.

This is the reciprocal of the proposition of Art. 4.

7. *Projective Aspect of Confocal Conics.*

Let p, q, r, s be the four given lines touched by each of the members of a four-line system of conics.

Let the line joining pq, rs be a
pr, sq be b
ps, qr be c.

Project pq and rs into the Circular Points at Infinity.

We thus get a system of confocal conics having as their four foci the points $p'r'$, $p's'$, $q'r'$, $q's'$ (Chap. II. Art. 17).

The Line at Infinity is a'.

The principal axes of the conics are b' and c'.

The centre is the point $b'c'$.

8. THEOREMS IN COAXAL CIRCLES.

Two and only two circles of a coaxal system touch a given line.

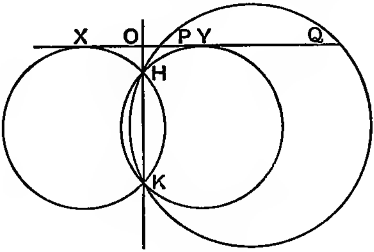

FIG. 51.

Any other circle of the system cuts the line in two points that harmonically separate the points of contact.(40)

This is plain since

$$\begin{aligned} OX^2 &= OH . OK, \\ OY^2 &= OH . OK, \\ OP . OQ &= OH . OK; \\ \therefore\ OX^2 = OY^2 &= OP . OQ. \end{aligned}$$

Hence [PXQY] is harmonic.

We can now generalise this Theorem by projection and obtain Desargues's Theorem.

DESARGUES'S THEOREM.

Two and only two conics of a Four-Point System touch a given line, and any other conic of the system cuts the line in two points that harmonically separate the points of contact.

RECIPROCAL OF DESARGUES'S THEOREM.

Two and only two conics of a Four-Line System pass through a given point, and the tangents to any other conic of the system from that point harmonically separate the tangents to the two conics that pass through that point.

9. *Any conic through the four points of intersection of two Rectangular Hyperbolas is a Rectangular Hyperbola.*

Let the two given Rectangular Hyperbolas S_1 and S_2 cut the Line at Infinity in H_1, K_1 and H_2, K_2 respectively.

Let any other conic S of the four-point system of conics defined by S_1 and S_2 cut the Line at Infinity in H, K.

Then the two conics of the four-point system that touch the Line at Infinity have for their points of contact two points that harmonically separate H_1, K_1 and H_2, K_2, by Chap. IV. Art. 8.

Hence, by Chap. II. Art. 11, these two points must be the Circular Points at Infinity I, J.

Consequently, H, K harmonically separate I, J, by Desargues's Theorem, and the conic S must therefore be a Rectangular Hyperbola.

Corollary I. *Any conic through the three vertices of a triangle and its orthocentre is a Rectangular Hyperbola.*

For A, B, C, P are the four intersections of the two Rectangular Hyperbolas AC, BP and BC, AP.

Corollary II. *Two Rectangular Hyperbolas cut in four points such that each is the orthocentre of the triangle formed by the other three.*

For, if two Rectangular Hyperbolas cut in the four points A, B, C, D, a degenerate conic of the four-point system thus defined is AD, BC, which is a Rectangular Hyperbola by the first part of the present article.

∴ AD is perpendicular to BC.

Similarly BD is perpendicular to CA.

Hence D is the orthocentre of the triangle ABC.

Corollary III. *Given four points* A, B, C, D *and a line l, we can project that line to infinity and at the same time the four points into other four points such that each is the orthocentre of the triangle formed by the other three.*

Let the two conics of the four-point system defined by A, B, C, D that touch l have as points of contact X, Y.

Project X, Y into the Circular Points at Infinity.

Then the four-point system become Rectangular Hyperbolas, by Chap. II. Art. 13.

Hence A', B', C', D' are such that each is the orthocentre of the triangle formed by the other three, by Cor. II.

10. *If the polars of two points* O_1 *and* O_2 *with respect to a given conic* S *cut* S *in* A_1, B_1 *and* A_2, B_2 *respectively, then* O_1, A_1, B_1, O_2, A_2, B_2 *all lie on the same conic.*

Project A_1, B_1 into the Circular Points at Infinity, and we have to shew that O_2', the two points of contact of the tangents, from O_2' to a circle, and the centre of that circle, are concyclic,—which is evident.

DUAL THEOREM.

A conic can be inscribed to the two triangles $O_1A_1B_1$ *and* $O_2A_2B_2$.

11. *The two tangents drawn from the orthocentre of a triangle to any parabola inscribed in the triangle are perpendicular.*

It is plain that all the parabolas inscribed to the triangle ABC form a four-line system, for they each touch the three sides of a given triangle and the Line at Infinity.

Hence the tangents from the orthocentre P form pairs of lines harmonically separating a certain pair of fixed lines through P (Chap. IV. Art. 8). It will therefore be sufficient to find the

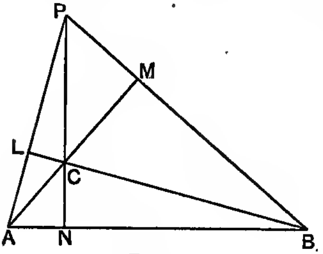

FIG. 52.

pair of lines that harmonically separate two of the above pairs of tangent lines (Chap. II. Art. 4).

Now, one conic of the system touching the three sides of the triangle ABC and the Line at Infinity is the Degenerate Point-Conic consisting of the point A and the point in which BC cuts the Line at Infinity. The tangents from P to this conic are PA and the line through P parallel to BC, and these tangents are perpendicular. Similarly the tangents from P to the point-conic consisting of B and the point in which CA meets the Line at Infinity are perpendicular. These two pairs of tangents are harmonically separated by PI and PJ, whence it is plain that all the pairs of tangents from P to the parabolas are harmonically separated by PI and PJ, *i.e.* are perpendicular.

COROLLARY. *If the pair of tangents from the orthocentre of a triangle to an inscribed conic be perpendicular, that conic is a parabola.*

12. THE POLAR CIRCLE. The Polar Circle of a triangle ABC is that circle which has the orthocentre P as centre, and with respect to which each side of ABC is the polar of the opposite vertex. In Fig. 52 $PL.PA=PM.PB=PN.PC=\rho^2$, where $\rho=$ the radius of the Polar Circle.(41)

Consider the following hypotheses:

I. Two triangles may have their six vertices all lying on the same conic.

II. Two triangles may have their six sides all touching the same conic.

III. Two triangles may be each self-conjugate with respect to the same conic.

If any one of the above hypotheses I., II., III., be true, the other two are also true.

The above propositions can all be very simply proved from the same figure (which involves only a triangle; its orthocentre and two perpendicular lines through it), and it will be sufficient to exemplify the demonstration of one of them.

To shew that if II. be true, I. and III. are also true.

Let the conic S be inscribed to the two given triangles $\mathsf{A}_1\mathsf{B}_1\mathsf{C}_1$ and $\mathsf{A}_2\mathsf{B}_2\mathsf{C}_2$. Project the line $\mathsf{A}_2\mathsf{B}_2$ into the Line at Infinity and C_2 into the orthocentre of $\mathsf{A}_1'\mathsf{B}_1'\mathsf{C}_1'$ (Chap. IV. Art. 9, Cor. III.).

Then S' is a parabola, since it touches the Line at Infinity $\mathsf{A}_2'\mathsf{B}_2'$. Also $\mathsf{C}_2'\mathsf{A}_2'$ and $\mathsf{C}_2'\mathsf{B}_2'$ are perpendicular (Chap. IV. Art. 11).

Hence a Rectangular Hyperbola passing through A_1', B_1', C_1', C_2', and A_2' will pass also through B_2', which proves I.

Also $\mathsf{C}_2'\mathsf{A}_2'$ and $\mathsf{C}_2'\mathsf{B}_2'$ being perpendicular are conjugate lines with respect to the Polar Circle of $\mathsf{A}_1'\mathsf{B}_1'\mathsf{C}_1'$, *i.e.* $\mathsf{A}_1'\mathsf{B}_1'\mathsf{C}_1'$ and $\mathsf{A}_2'\mathsf{B}_2'\mathsf{C}_2'$ are each self-conjugate triangles with respect to the above Polar Circle, which proves III.

These propositions may also be stated from another point of view, which owes its importance to the geometrical interpretation of Invariants.

If two conics be such that one triangle can be inscribed in the one and circumscribed to the other, then an infinite number of such triangles can be so described.

If two conics be such that one triangle can be *inscribed in the one self-conjugate to the other, then an infinite number of such triangles can be so described.*

If two conics be such that one triangle can be described about the one self-conjugate to the other, then an infinite number of such triangles can be so described.

The same figure suffices as before, and we shall exemplify the proof of one of them, *e.g.* the third proposition.

Let the triangle $A_1B_1C_1$ be described about the conic S_1, and be self-conjugate to the conic S_2.

Take any line A_2B_2 touching S_1, and let C_2 be its pole with respect to S_2. Let the tangents from C_2 to S_1 be C_2A_2, C_2B_2.

Then $A_2B_2C_2$ is described about S_1, and we wish to prove that it is at the same time self-conjugate to S_2.

Project A_2B_2 into the Line at Infinity and C_2 into the orthocentre of $A'_1B'_1C'_1$.

Then S'_1 is a parabola inscribed in $A'_1B'_1C'_1$, and since C'_2 is the orthocentre, the tangents $C'_2A'_2$ and $C'_2B'_2$ are perpendicular by Chap. IV. Art. 11.

Now, the conic S'_2 is self-conjugate to $A'_1B'_1C'_1$ and has C'_2 as the pole of $A'_2B'_2$, *i.e.* as the centre. S'_2 must therefore be the Polar Circle.

Furthermore, since $C'_2A'_2$ and $C'_2B'_2$ are perpendicular, therefore $A'_2B'_2C'_2$ is self-conjugate with respect to S'_2. The proposition is thus evident.

N.B. **The proposition that if two triangles be inscribed in one conic, they are described about another is merely the generalised form of the fact that if a triangle be**

described about a parabola, its Circum-Circle passes through the focus.(42)

13. *If two triangles* $A_1B_1C_1$ *and* $A_2B_2C_2$ *be such that* A_1 *is the pole of* B_2C_2, B_1 *of* C_2A_2, C_1 *of* A_2B_2 *with respect to a given conic* S; *then* (1) A_2 *is the pole* B_1C_1, B_2 *of* C_1A_1, C_2 *of* A_1B_1, *and* (2) *the two triangles* $A_1B_1C_1$ *and* $A_2B_2C_2$ *are in perspective.*

To prove (1) we note that since the polars of B_1 and C_1 each pass through A_2, therefore A_2 is the pole of the line B_1C_1, and so on.

To prove (2), let B_1C_1 meet B_2C_2 in L.
„ C_1A_1 „ C_2A_2 „ M.
„ A_1B_1 „ A_2B_2 „ N.

Project the points in which LM meets S into the Circular Points at Infinity. ....(1)

Let O' be the centre of the circle S'.

Since A_1' is the pole of $B_2'C_2'$, ∴ $O'A_1'$ is perpendicular to $B_2'C_2'$. ....(2)

Since A_2' is the pole of $B_1'C_1'$, ∴ $O'A_2'$ is perpendicular to $B_1'C_1'$. ....(3)

But $B_1'C_1'$ and $B_2'C_2'$ are parallel; ∴ A_1', O', A_2' are collinear. ....(4)

Similarly B_1', O', B' are collinear. ....(5)

It is now plain from (2), (3), (4), (5) that O' is the orthocentre of both triangles $A_1'B_1'C_1'$ and $A_2'B_2'C_2'$.

$$\left.\begin{array}{ll}\text{Hence} & O'C_1' \text{ is perpendicular to } A_1'B_1' \\ \text{and} & O'C_2' \text{ is perpendicular to } A_2'B_2'.\end{array}\right\} \ldots\ldots\ldots\ldots\ldots(6)$$

But since C_1' is the pole of $A_2'B_2'$ (by hypothesis),

∴ $O'C_1'$ is perpendicular to $A_2'B_2'$. ....(7)

Hence, by (6) and (7), $A_1'B_1'$ is parallel to $A_2'B_2'$, whence the proposition is evident.

COROLLARY. *Conversely, if two triangles* $A_1B_1C_1$ *and* $A_2B_2C_2$ *be such that the lines joining corresponding vertices are concurrent and meet in* O, *then there exists a conic such that* A_1 *is the Pole of* B_2C_2, B_1 *of* C_2A_2, C_1 *of* A_2B_2.

Using the notation of the foregoing proposition, project the line LMN into the Line at Infinity and at the same time the point O into the orthocentre of the triangle $A_1'B_1'C_1'$ (Chap. IV. Art. 9, Cor. III.).

We thus get the following figure:

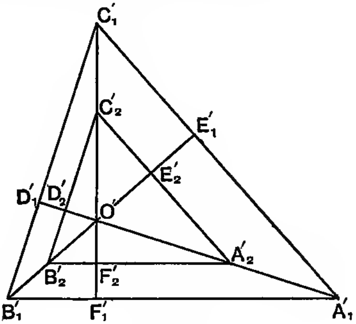

FIG. 53.

It can easily be seen that

$$O'A_1' \cdot O'D_2' = O'B_1' \cdot O'E_2' = O'C_1' \cdot O'F_2'.$$

Take each $= -\rho^2$, and it is plain that each vertex is the pole of the opposite side of the other triangle with respect to the circle, centre O' and radius ρ.

14. *Pascal's Theorem.*

Let ABCDEF be six points situated on a conic.

Then we can consider one of them, say A, fixed, and the order of the others permuted in $\frac{1}{2}\,\underline{|5}$, *i.e.* 60 ways.

Take one permutation of the letters and consider them as arranged round the circumference of a circle thus:

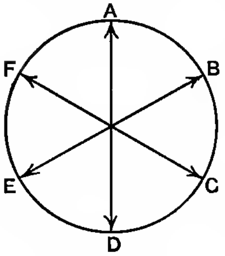

Fig. 54.

Let AB meet DE in L;
„ BC „ EF „ M;
„ CD „ FA „ N.

We wish to prove that L, M, N are collinear.

Project the points in which LM meets the given conic into the Circular Points at Infinity.

Then A′B′ will be parallel to D′E′.
„ B′C′ „ „ „ E′F′.

We wish to prove that C′D′ is parallel to F′A′.

Since A′B′ is parallel to D′E′, ∴ arc B′D′ = arc E′A′. (1)

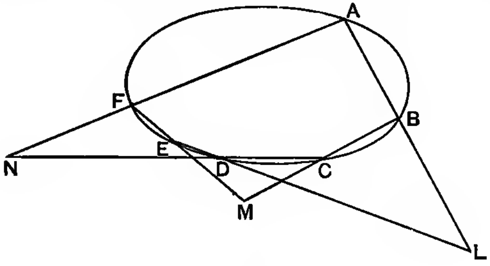

Fig. 55.

Since B′C′ is parallel to E′F′, ∴ arc F′B′ = arc C′E′. (2)

Adding (1) and (2), we get that arc F′D′ = arc C′A′. (3)

∴ F′A′ and C′D′ are parallel, *i.e.* L′M′N′ are collinear.

Corollary. We might take another permutation of the letters **ABCDEF** and arrange them as shewn on the subjoined figure:

By joining them up according to the arrangement on Fig. 56, we should find that the intersections of

AE and **CF**,
EB and **FD**,
BC and **DA**

are also collinear.

Proceeding in this way, we should get 60 lines of collineation corresponding to the 60 possible arrangements of the given six letters explained at the beginning of this article. These 60 lines have remarkable properties, which are explained at length in the Appendix to Salmon's Conics.

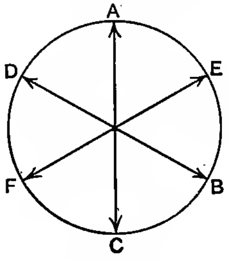

Fig. 56.

15. *Brianchon's Theorem.*

Brianchon's Theorem is the reciprocal of Pascal's Theorem, the conic itself being chosen as the Base-Conic of the reciprocation, and may be enunciated thus:

If **ABCDEF** *be six given points on a conic, and if the tangents at* **A**, *etc., be a, etc., then the lines joining the points*

ab and *de*
bc and *ef*
cd and *fa*

are concurrent.

Project the points of intersection of the line **LMN** of last article into the Circular Points at Infinity.

Then, since A′B′ and D′E′ are parallel, it is plain that the line joining the points *a′b′* and *d′e′* passes through the centre. So for the lines joining *b′c′* and *e′f′*; *c′d′* and *f′a′*. .

Hence the theorem enunciated is evident.

Corollary. *For any given hexagon inscribed in a conic, a Pascal Line is the polar of a Brianchon Point.*

16. The following propositions are very important and can be easily proved by projecting the points where one of the diagonals of the inscribed quadrangle meets the conic into the Circular Points at Infinity.

We shall content ourselves with merely enunciating them.

Theorem 1.

If a quadrangle be inscribed in a conic, and at its angular points four tangents be drawn, the six points of intersection of these four tangents lie in pairs on the sides of the harmonic triangle of the inscribed quadrangle.

Theorem 2.

If a quadrangle be inscribed in a conic and tangents be drawn at its angular points forming a circumscribed quadrilateral, the diagonals of the quadrangle and the quadrilateral are concurrent and form a harmonic pencil.

Theorem 3.

If tangents be drawn at the angular points of an inscribed quadrangle, forming a circumscribed quad-

rilateral, the third diagonals of the quadrangle and quadrilateral are coincident, and the extremities of the one are harmonic conjugates with respect to the extremities of the other.

17. *The Φ-Conic.*

We shall now require the following theorem in Modern Geometry:

If A and B be two fixed circles, and if a variable point P move so that the tangent from P to the circle A bears a constant ratio to the tangent from P to the circle B, then will P describe a circle coaxal with A and B.(43)

If S_1 *and* S_2 *be two given conics, the envelope of a line p cutting* S_1 *and* S_2 *such that the intersections of p with* S_1 *harmonically separate the intersections of p with* S_2*, is a conic called the Φ-Conic.*

Project two of the intersections of S_1 and S_2 into the Circular Points at Infinity.

Let A' and B' be the centres of the circles S_1' and S_2' thus obtained.

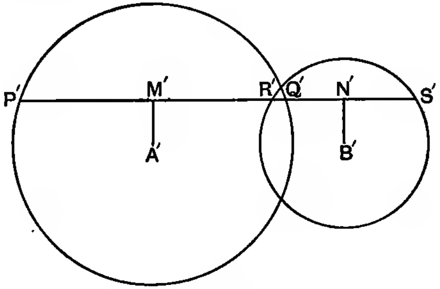

Fig. 57.

Let P', Q' harmonically separate R', S'.

Drop the perpendiculars $A'M'$ and $B'N'$ on to the line p'. Then M' is the mid-point of $P'Q'$ and N' is the mid-point of $R'S'$.

Now, by an elementary property of Harmonic Division, since P', Q' harmonically separate R', S' and since M' is the mid-point of $P'Q'$,

$$\therefore\ M'Q'^2 = M'R' . M'S',$$

$$i.e.\quad \frac{M'P' . M'Q'}{M'R' . M'S'} = -1.$$

But $M'R' . M'S'$ = the square of the tangent from M' to the circle whose centre is B'.

Also $M'P' . M'Q'$ = the square of the tangent (imaginary in the above figure) from M' to the circle whose centre is A'.

Hence the line p' moves so that the tangent from M' to the circle A' bears to the tangent from M' to the circle B' the constant ratio $\sqrt{-1}$.

Thus, by the proposition quoted at the beginning of the present article, M' describes a circle; and by the proposition quoted in Chap. III. Art. 25, $M'N'$ envelopes a conic having A' as focus. Similarly we can shew that $M'N'$ envelopes a conic having B' as focus, *i.e.* this Conic-Envelope touches $A'I'$, $A'J'$, $B'I'$, $B'J'$ (the tangents at I' and J' of the given circles).

Hence in the original figure p envelopes a conic (called the Φ-Conic) touching the tangents to S_1 *and* S_2 *at the four points of intersection of* S_1 *and* S_2.

18. *The* F-*Conic.*

The Dual Theorem to that proved in last article is:

If a point P *move so that the tangents from* P *to a Conic* Σ_1 *harmonically separate the tangents from* P *to a conic* Σ_2, *then* P *describes a conic (called the* F-*Conic), which passes through the eight points of contact of the four common tangents of* Σ_1 *and* Σ_2.

19. *Properties of a System of Four-Point Conics.*

Theorem 1.

One and only one conic of the system is in general a Rectangular Hyperbola.

This follows by using Desargues's Theorem and Chap. II. Art. 4.

Theorem 2.

Two conics of the system are in general parabolas.

This is Desargues's Theorem.

Theorem 3.

The polars of a given point with respect to a system of Four-Point Conics all pass through another given point.

This is proved in books on Modern Geometry in the case of Coaxal Circles.

N.B. Let P be a given point and let Q be the point through which all the polars of P pass. Consider the line PQ. All the conics of the system harmonically separate P, Q. *Hence* P *and* Q *are the points of contact of the two conics of the Four-Point System that touch the line* PQ (see Chap. IV. Art. 8).

Theorem 4.

The Eleven-Point Conic with respect to a given line.(44)

The locus of the poles of a given line p with respect to a Four-Point System of Conics is a conic (called the Eleven-Point Conic with respect to the given line).

Let all the conics of the system pass through the four points P, Q, R, S.

Let the two conics of the system that touch the given line *p* touch it at H and K.

Project H and K into the Circular Points at Infinity.

We then have to find the locus of the centres of a series of Rectangular Hyperbolas passing through four points P′, Q′, R′, S′ (Chap. IV. Art. 9, Cor. III.), each of which is the orthocentre of the triangle formed by the other three.

Let F′ be the middle point of Q′R′ and let W′ be the middle point of P′S′.

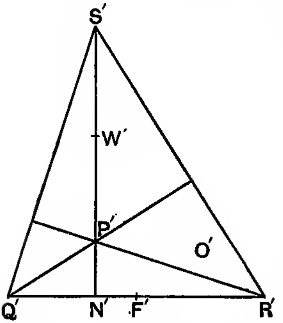

Fig. 58.

Let O′ be the centre of one of the system of Rectangular Hyperbolas passing through P′, Q′, R′, S′.

Now, since S′P′ and Q′R′ are two perpendicular chords of this Rectangular Hyperbola, and since the angle between two chords of a Rectangular Hyperbola is equal or supplementary to the angle between their conjugate diameters, therefore O′W′ is perpendicular to O′F′.

Hence O′ describes the circle on W′F′ as diameter; which circle plainly passes through N′, and is therefore the Nine-Points Circle.

Thus, in the original figure, the conic which is the locus of the poles of the given line passes through the following *Eleven* points (and hence derives its name):

(1) The three vertices of the Harmonic Triangle of the given four points PQRS.

(2) The Harmonic Conjugate of the point in which PQ cuts p, with respect to P and Q, and the other five points similarly determined.

(3) The two points on p in which this line is touched by the two members of the system that touch p.

20. *Properties of a System of Four-Line Conics.*

Theorem 1.

In general one and only one conic of the system is a parabola.

For only one conic touches 5 given lines.

Theorem 2.

In general two and only two conics of the system are Rectangular Hyperbolas.

This is proved under Theorem 7 of the present article.

Theorem 3.

The Director-Circles of a System of Four-Line Conics form a coaxal system.

Let Σ_1 and Σ_2 be two conics of the system and let their Director-Circles meet in H and K.

Consider H.

By Chap. IV. Art. 8 two and only two conics of the system pass through H, and the tangents to any other conic of the system from H harmonically separate the tangents to the two conics that pass through H.

Consider the tangents from H to Σ_1 and denote them by $H\Sigma_1$. Since H lies on the Director-Circle of Σ_1, the tangents $H\Sigma_1$ are perpendicular.

Hence the tangents $H\Sigma_1$ harmonically separate HI and HJ.

Similarly the tangents $H\Sigma_2$ harmonically separate HI and HJ.

Thus HI and HJ will be respectively the tangents at H to the two conics of the system that pass through H.

Let Σ_3 be another conic of the system.

Then $H\Sigma_3$ harmonically separate HI and HJ by the proposition quoted above, and are therefore perpendicular.

We thus see that H lies on the Director-Circle of Σ_3. Similarly K lies on the Director-Circle of Σ_3.

Hence the Director-Circles of the Four-Line System of Conics form a coaxal system.

Theorem 4.

The Centres of all Conics of a Four-Line System are Collinear.

For their Director-Circles form a Coaxal System, whose centres are collinear, and the centre of a conic is the same as that of its Director-Circle.

Theorem 5.

The mid-points of the diagonals of a Complete Quadrilateral are collinear.

Let the Complete Quadrilateral be formed of the four lines p, q, r, s.

Then one degenerate conic of the Four-Line System is composed of the two points pq and rs.

The centre of the degenerate conic, composed of pq

and *rs*, is obviously the middle point of the line joining these two points. Hence the line which is the locus of the centres of all conics of the Four-Line System passes through the mid-point of the diagonal joining *pq* and *rs*. Similarly it passes through the middle points of the lines joining *pr* and *sq*; *ps* and *qr*.

Theorem 6.

The common Radical Axis of the above system of Coaxal Director-Circles is the directrix of the parabola of the Four-Line System.

For in the case of a parabola its "Director-Circle" degenerates into the conic made up of its Directrix and the Line at Infinity.

The Directrix of the Parabola of the Four-Line System must therefore pass through H and K.

Theorem 7.

The Limiting-Points of the above system of Coaxal Director-Circles are the centres of the two Rectangular Hyperbolas of the System.

For the Limiting-Points of a Coaxal System of Circles may be regarded as those two circles of the system whose radii are infinitesimal. But the Director-Circle of a Rectangular Hyperbola is the infinitesimal circle at its centre, whence the theorem just enunciated follows.

Theorem 8.

The three circles described on the lines joining the points

pq and rs,
pr and sq,
ps and qr,

as diameters, belong to the above System of Coaxal Director-Circles.

For plainly they are respectively the Director-Circles of the degenerate Point-Conics composed of the points

pq and rs,
pr and sq,
ps and qr.

Theorem 9.

The Circum-Circle of the Diagonal Triangle abc cuts all the members of the above system of Coaxal Director-Circles orthogonally

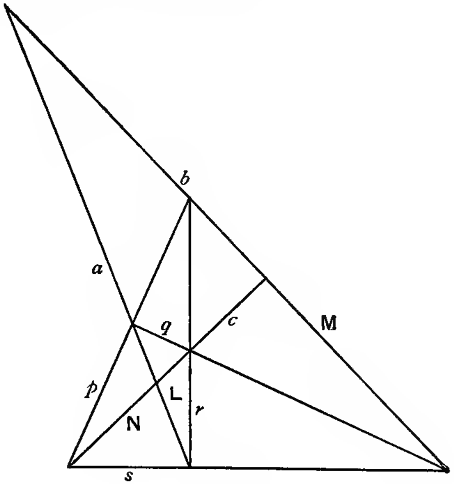

FIG. 59.

For let L be the middle point of the line joining *pq*, *rs*
and „ M „ „ „ „ *pr*, *sq*
and „ N „ „ „ „ *ps*, *qr*.

By Chap. I. Art. 13 *ab* and *ca* harmonically separate *pq* and *rs*; hence *pq* and *rs* are conjugate points with respect to the circle circumscribing *abc*.

Hence the circle on *pq* and *rs* as diameter cuts orthogonally the Circum-Circle of the triangle *abc*.

Similarly the Circum-Circle of *abc* cuts orthogonally the circles on *pr* and *sq*, and *ps* and *qr* as diameters.

But when a circle cuts two circles of a Coaxal System orthogonally, it cuts all the members of the Coaxal System orthogonally, wherefore the Circum-Circle of the triangle *abc* cuts orthogonally all the members of the above system of Coaxal Director-Circles.

Theorem 10.

The Circum-Circle of the triangle abc cuts the line joining the middle points of the diagonals in the Limiting-Points of the System.

For by an elementary property of Coaxal Circles, any circle cutting orthogonally a system of Coaxal Circles must cut orthogonally the Point-Circles of the system, *i.e.* must pass through the Limiting-Points which constitute the Point-Circles of the system.

21. Many properties of conics belong to the type dealt with at the end of Art. 12 of Chap. IV. The following is an important additional example:

If one quadrilateral can be circumscribed to the conic Σ *and inscribed in the conic* S, *an infinite number of such quadrilaterals can be so described.*

Let the points P, Q, R, S lie on the conic S.

Let PQ, QR, RS, SP touch the conic Σ.

Let PQ and RS meet in L,
and let QR and SP meet in M.

Project the points in which the line LM meets the conic S into the Circular Points at Infinity.

We thus get a circle S′, in which is inscribed a rectangle P′Q′R′S′ whose sides touch the conic Σ′.

Thus S′ will be the Director-Circle of the conic Σ′.

It is thus plain that an infinite number of rectangles can be inscribed to the circle S′ and circumscribed to the conic Σ′, whence the theorem enunciated follows.

EXAMPLES IV.

1. The opposite sides of the hexagon ABCDEF are parallel and the diagonal CF is parallel to the sides AB and DE. BC and AF intersect in P; CD and EF in Q; BD and AE in R. Shew that P, Q, R are in one straight line. (Math. Trip.)

2. A system of conics has a common focus and two fixed tangents. Shew that the locus of their centres is a straight line. (Clare, etc.)

3. Through the extremities of any two focal chords of an ellipse a conic is described; if this conic pass through the centre of the ellipse, it will cut the major axis in another fixed point.

4. A system of rectangular hyperbolas have a common centre O, and are such that two fixed points A and B are conjugate with respect to each of them. Shew that the hyberbolas pass through four fixed points which lie on the external and internal bisectors of the angle AOB. (Clare, etc.)

5. The conic which touches the asymptotes of a conic and also two tangents and their chord of contact is a parabola. (Clare, etc.)

6. If two conics with a common directrix meet in four points, these four points lie on a circle, whose centre is on the straight line joining the corresponding foci.

7. A given pair of points and a given pair of lines are each conjugate with respect to a circle; shew that the locus of the centre of the circle is a conic whose asymptotes are perpendicular to the given lines. (St. John's.)

8. A variable circle is drawn through the foci of an ellipse; find the locus of the point of contact with the circle of a common tangent to it and the ellipse.

9. ABCD are four points on a circle of which O is the centre. Shew that the polars of O, with respect to a system of conics through ABCD, are parallel. (Clare, etc.)

10. The polars of two points T_1, T_2, with respect to a parabola, meet the curve in P_1, Q_1, P_2, Q_2. Prove that if the points T_1, T_2, P_1, Q_1, P_2, Q_2 lie on a circle, the focus must lie at the middle point of T_1T_2. (Clare, etc.)

11. If two circles cut orthogonally, prove that their Φ-conic degenerates into the point-conic composed of their two centres.

12. ABCD are four given points on a straight line such that a parabola can be described passing through C and D, and having A on its axis and B on the tangent at its vertex; shew that there are an infinite number of such parabolas, and that the locus of the points of contact of tangents drawn parallel to AD is a circle. (Clare, etc.)

13. Prove that, if the axes of two parabolas be at right angles, the circle round the triangle formed by the three common tangents at a finite distance is the circle on the line joining the foci as diameter. (Clare, etc.)

14. PQ are two points on an ellipse; the tangents at P and Q meet in T, and PQ meets the directrices in R, R′; SR meets TP in V and HR′ meets TQ in V′; shew that VV′, SQ, HP are concurrent. (Clare, etc.)

15. Prove (from the consideration that one and only one parabola touches four given straight lines, or otherwise) that the circum-circles of the four triangles formed by four straight lines have a common point.

16. Prove also that the orthocentres of the four triangles in last question lie on a straight line.

17. AA′, BB′, CC′ are six points forming the vertices of a complete quadrilateral. Points $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are taken on AA′, BB′, CC′ respectively such that

$$[\mathrm{A}\alpha\mathrm{A}'\alpha'] = \mathrm{B}\beta\mathrm{B}'\beta'] = [\mathrm{C}\gamma\mathrm{C}'\gamma'].$$

Shew that the six points $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ lie on a conic section. (Clare, etc.)

18. Given a triangle and two points, prove that we can project the triangle so as to have one of these points for the centroid and the other for the orthocentre.

19. Two points O, O′ are taken within a triangle ABC; lines drawn from the angular points to O and O′ (and produced if necessary) determine on the sides point-pairs XX′, YY′, ZZ′ respectively. Corresponding sides of the triangles XYZ, X′Y′Z′ meet in P, Q, R; shew that the six points X, Y, Z, X′, Y′, Z′ lie on a conic of which PQR is a self-conjugate triangle. (Math. Trip.)

20. Two circles have each double contact with an ellipse and the chords of contact intersect in a point P. Shew that P is one of the Limiting-Points of the coaxal system to which the circles belong, and determine the other.

(Math. Trip.)

21. If the centre of a rectangular hyperbola H lies on the director-circle of a conic C, prove that the locus of points whose tangents to C, H form a harmonic pencil is a rectangular hyperbola. (St. John's.)

22. Two pairs of parallel tangents are drawn to a conic, and parallels to them are drawn through a focus; shew that the points where they meet the tangents lie on a circle.

23. Shew that, if the pole of one common chord of a circle and a fixed conic lie on the circle, the pole of the opposite common chord lies on the circle; also shew that the lines joining these poles to the centre of the conic make equal angles with the major axis of the conic.

24. Two conics have a common directrix: prove that their common points lie on a parabola which has the directrix as a diameter, the tangent at the extremity of which bisects the line joining the foci of the conics. (Jesus, etc.)

25. S and S′ are the foci of a hyperbola, LPM a tangent meeting the asymptotes at L and M. Prove that a circle will go round SLS′M. (St. Catharine's.)

26. P is a point on a rectangular hyperbola whose centre is C, and a line is drawn through C perpendicular to CP. Through Q, any point on the curve, lines are drawn parallel to the asymptotes meeting this line in L, M. Shew that LPM is a right angle. (Clare, etc.)

27. Prove that the locus of the centres of conics having double contact with a given conic at the extremities of a given line is the diameter of the given conic conjugate to the given line. (Trinity.)

28. If two rectangular hyperbolas intersect in four real points, prove that the middle points of the six common chords lie on a circle. (St. Catharine's.)

29. PP′ is a diameter of a rectangular hyperbola, and Q, Q′ are any two points on the hyperbola; prove that the angle between PQ and P′Q′ is equal to the angle between PQ′ and P′Q. (Corpus, etc.)

30. In a rectangular hyperbola conjugate diameters are equally inclined to each asymptote.

31. Prove that the locus of the pole of a given straight line with reference to a series of confocal conics is a straight line.

32. Conics are drawn having double contact with each of a pair of similar and similarly situated conics. Prove that the locus of their centres consists of a conic similar and similarly situated to the given conics and of a straight line through their centres. (Trinity.)

33. If the line joining two conjugate points with respect to a circle subtends a constant angle at the centre and one point moves on a straight line, find the locus of the other. (Peterhouse, etc.)

34. Two tangents to a conic from a point T meet a given straight line in P, Q, and PQ subtends a right angle at a given point. Shew that T traces out a conic section. (Clare, etc.)

35. If DEF be the pedal triangle of the triangle ABC, shew that the triangle DEF is a self-conjugate triangle relative to any rectangular hyperbola circumscribing the triangle ABC. (Queens'.)

36. The circles described on the diagonals of a complete quadrilateral as diameters may all touch at a single point, and this point is the point of contact of the circle circumscribing the diagonal triangle with the line bisecting the three diagonals, which is in this instance a tangent to this last circle. (Math. Trip.)

37. A line is drawn through the vertex in any given direction cutting the parabola in B, and with B as vertex a parabola is drawn to pass through the vertex and with axis parallel to the axis of the first parabola. Lines are drawn parallel to the axis and intercepted between the two parabolas. Shew that the middle points of these intercepts lie on a given line. (Magdalene.)

38. Two points P, Q are taken on a rectangular hyperbola whose centre is K. Lengths FG, FH are measured along PQ from its middle point F in opposite directions, each = KF. Prove that the asymptotes are KG, KH. (Math. Trip.)

39. Two tangents to a circle are fixed, two others are drawn so as to form with the two fixed tangents a quadrilateral having two opposite sides along the fixed tangents; shew that the locus of intersection of diagonals of this quadrilateral is a straight line, and find its position. (Queens'.)

40. A system of ellipses is drawn, each of which touches a hyperbola and has its asymptotes for conjugate diameters; prove that if the pole of a given line with respect to any one of the ellipses lies on the hyperbola, then its pole with respect to any other also lies on the hyperbola. (Peterhouse, etc.)

41. The locus of the pole of a given straight line with respect to a system of circles having a common radical axis is a hyperbola, one of whose asymptotes is parallel to the common radical axis and the other perpendicular to the given line. (Pembroke, etc.)

42. If ellipses be inscribed in a rectangle, prove that the foci lie on a rectangular hyperbola passing through the corners. (St. Catharine's.)

43. If the angular points B and C of a given triangle ABC move upon two straight lines OX and OY, shew that on each of the sides AB and AC, produced if necessary, one other point can be found which moves on a straight line passing through O. (King's.)

44. Parallel tangents are drawn to a system of circles which all pass through two fixed points. Prove that the points of contact lie on a rectangular hyperbola. (Sidney.)

45. Two triangles ABC, PQR are in perspective; so that the lines AP, BQ, CR meet in S, and the points X, Y, Z (X is

the intersection of BC, QR, etc.) are collinear. Prove that in the resulting figure, composed of ten points and ten lines, the points and lines can be paired in such a way that each point is the pole of the corresponding line with respect to a conic. (St. John's.)

46. Prove that there are two points at which three circles subtend the same angle and that the line joining them passes through the point from which the tangents to the circles are equal. (Magdalene.)

47. Shew that the circles whose diameters are chords of a rectangular hyperbola drawn parallel to a given direction constitute a coaxal system; and that the systems corresponding to two directions at right angles are orthogonal to one another. (St. John's.)

48. P is any point on a hyperbola whose foci are S and H. If with P and S for foci a hyperbola is constructed passing through H; and with P and H for foci another is drawn through S, prove that these intersect on the original curve. (Clare, etc.)

49. A and B are two fixed points. AP, BP meet a fixed line which is not parallel to AB in two points Y and Z. If YZ be constant in length and in direction, shew that the locus of P is a hyperbola. (Clare, etc.)

50. In a complete quadrilateral, if AA′, BB′, CC′ be the three diagonals, and A, A′; B, B′ pairs of conjugate points relative to a circle, then shew that C, C′ will be conjugate points relative to the same circle. Shew also that this circle cuts at right angles the circles on AA′, BB′, CC′ as diameters, and further the circle circumscribing the triangle formed by the lines AA′, BB′, CC′. (St. John's.)

51. From a point P tangents PQ, PR are drawn to a hyperbola, the directrices of which meet its transverse axis in X, X′. Prove that if the tangents separate PX, PX′

harmonically, the locus of P is a hyperbola; and account for the positions of its vertices and intersections with the given hyperbola. (St. John's.)

52. If the extremities of each of two diagonals of a complete quadrilateral are conjugate points with respect to a given conic, the extremities of the third diagonal also will be conjugate points with respect to the same conic.

53. If two pairs of opposite sides of a complete quadrangle are conjugate lines with respect to a conic, the two remaining sides are also conjugate lines with respect to the same conic.

54. If four circles are described to touch every three sides of a quadrilateral, shew that their centres are concyclic.

55. A circle circumscribes a quadrilateral formed by four tangents to a conic; shew that if it passes through one focus it also passes through the other. (Clare, etc.)

56. A variable circle touches two fixed parallel lines tangent to a given conic. Shew that the locus of intersection of the other two common tangents is a conic confocal with the given conic. (Clare, etc.)

57. AB, CD are the common tangents of two circles which cut at right angles. Prove that if any point is taken on either of the chords of contact AC or BD the four tangents drawn from it to the circles form a harmonic pencil. (St. John's.)

58. Two points P and Q are taken outside a conic whose centre is C, such that the principal axes of the conic bisect the angle PCQ Prove that the four tangents from P and Q to the conic touch also another conic of which C is a focus. (Clare, etc.)

59. Four intersecting straight lines are drawn in a plane. Prove that the circum-circles of the triangles formed by the four lines are concurrent at a point which is concyclic with their four centres.

60. The tangent at a fixed point P of an ellipse whose foci are S and S' meets a pair of conjugate diameters in the points T and T. Shew that the locus of the other intersection of the circles SPT, $S'PT'$ is a circle. (Math. Trip.)

61. The triangles $A_1B_1C_1$, $A_2B_2C_2$ are reciprocal with respect to a given circle; B_2C_2, C_1A_1 intersect in P_1 and B_1C_1, C_2A_2 intersect in P_2. Shew that the radical axis of the circles which circumscribe the triangles $P_1A_1B_2$, $P_2A_2B_1$ passes through the centre of the given circle. (Math. Trip.)

62. Of the two tangents drawn from P to a variable circle, one is parallel to a given tangent and the other cuts a second given tangent in a fixed point. Shew that the locus of P consists of two confocal parabolas. (Math. Trip.)

63. A circle is drawn touching a given line at a given point, and is touched by a second circle which also touches a second given line at a given point; prove that the locus of the point of contact of the two circles consists of two circles. (King's, etc.)

64. If two conics cut orthogonally at their four points of intersection, shew that these are concyclic.

65. If two conics have double contact, then every conic confocal to the first has double contact with some one confocal to the second, and the four common tangents to any confocal to the first and any confocal to the second touch a circle. (Math. Trip.)

66. Having given a directrix of a conic and the eccentricity and a point on the curve, shew that the conic always touches a fixed conic having the same eccentricity.

(Caius, etc.)

67. Circles are described through two fixed points A, B and a point P which always lies on a given line l. Prove that the other end of the diameter through P lies on a fixed hyperbola, of which the asymptotes are perpendicular to AB and l. (Trinity.)

68. If any straight line PQRS be drawn cutting two given circles in P, S and Q, R; the segments PQ, RS subtend equal angles at either of the limiting-points of the coaxal system to which the given circles belong. (Trinity.)

69. Two conics have a common focus and their axes are in different directions. One of the chords through the common focus meets the conics in P and Q, these points being both on the same side of the focus. Prove that as the direction of the chord changes, the locus of the point of intersection of tangents at P and Q is a straight line which makes angles with the axes of the two conics such that their cosines are inversely proportional to their respective eccentricities.

70. A circle C is reciprocated with regard to another circle C′ into a conic S. Shew that C and S are also reciprocal with regard to a rectangular hyperbola concentric with and touching C′. Shew also that, when the circles do not intersect, C and S are reciprocal with respect to another real hyperbola having double contact with C′; the polar with regard to C′ of the external limiting-point being the chord of contact, and the pole of the radical axis being the centre. (Math. Trip.)

71. Two concentric ellipses S, S′ are such that parallelograms inscribed in S may be also circumscribed about S′. Shew that the ellipses have one real parallel pair of external common chords; and that, if through the poles of these chords with respect to S′ lines be drawn parallel to the sides of the variable parallelogram, the locus of the eight points of intersection is a conic concentric with and similar to S. (Math. Trip.)

72. Two conics are such that a quadrilateral can be inscribed to the first and circumscribed to the second. Shew that the two lines joining the points of contact of the inner conic harmonically divide one or other of the common chords of the conics.

73. Let there be given a line L and a conic S: upon L take the four points a, b, c, d such that $ab = bc = cd$ and upon S take any point P: let Pa, Pb, Pc, Pd cut S in A, B, C, D respectively. Prove that, if P′ be any other point of S, a line L′ can be found which will meet P′A, P′B, P′C, P′D in four points a', b', c', d' such that $a'b' = b'c' = c'd'$. (Clare, etc.)

74. Prove that the locus of the foci of conics touching four given straight lines is a curve of the third degree, which reduces to a rectangular hyperbola when the four lines form a parallelogram. (Queens'.)

75. A, B, C, D are four points in a plane, no three of which are collinear, and a projective transformation interchanges A and B and also C and D. Give a pencil and ruler construction for the point into which any arbitrary point P is changed; and shew that any conic through ABCD is transformed into itself. (Math. Trip.)

76. ABCD are four points in a plane, no three of which are collinear. Shew that there is one projective transformation in which (A, P), (B, Q), (C, R), (D, S) are corresponding points. When the planes coincide, how many points coincide with their corresponding points?

If ABCD is a square, and P, R are the circular points, shew that the circle ABCD transforms into the circle on QS as diameter. (St. John's.)

77. Shew that there are two distinct kinds of point-to-point correspondence in a plane such that any two corresponding triangles ABC, A′B′C′ are similar, and that one of either kind is determined by the assignment of two pairs of corresponding points. Investigate the number and position of the points that correspond to themselves. (St. John's.)

78. Shew that the six points of intersection of three circles cannot lie on a conic except when the circles have their centres collinear. (Clare, etc.)

79. ABCD is a parallelogram, and a conic is described to touch its four sides. If S is a focus of this conic, and if with S as focus a parabola is described to touch AB, BC, prove that the axis of the parabola passes through D. (Pembroke, etc.)

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